Necessary and Sufficient Conditions on Measurements of Quantum Channels

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Quantum supermaps are a higher-order generalization of quantum maps, taking quantum maps to quantum maps. It is known that any completely positive, trace non-increasing (CPTNI) map can be performed as part of a quantum measurement. By providing an explicit counterexample we show that, remarkably, not every quantum supermap sending a quantum channel to a CPTNI map can be realized in a measurement on quantum channels. We find that the supermaps that can be implemented in this way are exactly those transforming quantum channels into CPTNI maps even when tensored with the identity supermap.

One of the most puzzling aspects of quantum mechanics has always been the need to consider probabilistic processes to describe the observation of physical systems. The development of quantum information theory has turned this puzzling feature into a resource for many protocols. Think, for instance, of the implementation of quantum computation through measurements (measurement-based quantum computation) [49, 50], of quantum cryptographic protocols [9, 42, 5, 53], or of the generation of random numbers [38].

Specializing our attention to finite-dimensional quantum systems, the most general quantum measuring device can be described by a set of linear maps that are completely positive and trace non-increasing (CPTNI). The maps in this set must sum to a quantum channel, namely to a completely positive and trace-preserving (CPTP) linear map [44, 55, 54]. This situation is described by a *quantum instrument* [29, 40, 1, 54], a quantum channel that takes a quantum system as input, and outputs a a classical-quantum system, where the classical system represents the "meter" read by the experimenter. From the classical outcome read on the meter, one can infer which CPTNI map occurred during the experiment.

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This characterization of quantum experiments, in conjunction with the fact that quantum channels with trivial (i.e. 1-dimensional) input represent states [13], singles out channels as the fundamental objects of quantum theory, encapsulating all the other processes.

For this reason, it is important to understand how to manipulate quantum channels. The study of such manipulations, initiated in [13, 14, 18], has both practical [34, 12, 13, 19, 48, 43, 20] and foundational consequences [56, 45, 18, 46, 43], and has led to the development of new research areas, such as resource theories of quantum processes [27, 8, 23, 31, 30, 32]. The manipulation of quantum channels is implemented by *supermaps* [13, 14, 18, 47, 10, 31], which are linear transformations sending linear maps to linear maps. In this setting, *superchannels* [13, 31] represent the way a channel can evolve deterministically, in the same way as channels represent the deterministic evolution of a quantum state. Superchannels are the supermaps that take quantum channels to quantum channels even when tensored with the identity supermap [18]. Measurements on channels are then described by a set of supermaps that sum to a superchannel, giving rise to the notion of a quantum *super-instrument*.

In this letter we focus on measurements performed on quantum channels, and we show that a naive application of a condition similar to CPTNI in quantum theory is *not* enough to single out physical supermaps, viz. those that can arise in an experiment performed on quantum channels.

We will adopt the following notation.

Notation. $\mathscr{B}(\mathscr{H})$ denotes the set of bounded linear operators on the finite-dimensional Hilbert space $\mathscr{H}$, $\mathscr{B}_h(\mathscr{H})$ the set of bounded hermitian operators on $\mathscr{H}$, and $\mathscr{D}(\mathscr{H})$ the set of density matrices on $\mathscr{H}$. Every letter without a subscript denotes a pair of systems $A := A_0A_1$, where A_0 is usually regarded as an input system, and A_1 as an output system. Thus $\Psi^A := \Psi^{A_0\to A_1}$ denotes a linear map with input A_0 and output A_1, and $\mathscr{L}^A := \mathscr{L}^{A_0\to A_1}$ is the set of such linear maps, from $\mathscr{B}\left(\mathscr{H}^{A_0}\right)$ to $\mathscr{B}\left(\mathscr{H}^{A_1}\right)$. $|A_0|$ denotes the dimension of $\mathscr{H}^{A_0}$. A supermap $\Theta^{A\to B}$ takes elements of $\mathscr{L}^A$ to elements of $\mathscr{L}^B$. Finally, a tilde over a system, like in $A_0\widetilde{A}_0$, indicates that we are considering two identical copies of a system (in this case A_0). We adopt the following convention concerning partial traces: if M^{AB} is a matrix on $A_0A_1B_0B_1$, M^{AB_0} denotes the partial trace on the missing system B_1: $M^{AB_0} := \mathrm{Tr}_{B_1}\left[M^{AB}\right]$. In summary, when a superscript is missing, we have taken the partial trace over the missing system of the original matrix.

The first condition one must require of physical supermaps is that they be *completely CP-preserving* (CPP): they should send CP maps to CP maps even when tensored with the identity supermap. In formula, a supermap $\Theta^{B\to C}$ is CPP if for all bipartite CP maps $\Psi^{AB} \in \mathscr{L}^{AB}$, we have

$$\left(\mathscr{I}^A \otimes \Theta^{B\to C}\right)\left[\Psi^{AB}\right] \geq 0, \tag{1}$$

where $\mathscr{I}^A := \mathscr{I}^{A\to A}$ is the identity supermap. This is analogous to the CP condition for quantum maps.

The second condition, analogous to being TNI for quantum maps, is that a physical supermap should send CPTNI maps to CPTNI maps. If a supermap is CPP, demanding this is equivalent to requiring that it should take *CPTP* maps to CPTNI supermaps (see Appendix A). More precisely, a supermap $\Theta^{B\to C}$ is *CPTNI-preserving* if it is CPP and

$$\mathrm{Tr}\left[\Theta^{B\to C}\left[\Psi^B\right]\left(\rho^{C_0}\right)\right] \leq 1, \tag{2}$$

for any CPTP map $\Psi^B \in \mathscr{L}^B$ and any $\rho^{C_0} \in \mathscr{D}\left(\mathscr{H}^{C_0}\right)$.

A measurement on quantum channels (called a *super-measurement*) is described by a set of CPTNI-preserving supermaps $\left\{\Theta_x^{B\to C}\right\}_{x\in X}$, indexed by the outcome x of the measurement, such that $\sum_{x\in X}\Theta_x^{B\to C}$ is a superchannel. This gives rise to the super-instrument

$$\Upsilon^{B\to X_1C}\left[\Psi^B\right] = \sum_{x\in X} |x\rangle\langle x|^{X_1} \otimes \Theta_x^{B\to C}\left[\Psi^B\right], \tag{3}$$

where system X_1 represents the classical meter and $\left\{|x\rangle^{X_1}\right\}$ is an orthonormal basis of X_1.

Our main result is that, surprisingly, *not all* CPTNI-preserving supermaps can arise in a quantum super-measurement, therefore *not all* CPTNI-preserving supermaps are physical. An example is the supermap $\Theta^{B\to C}$ whose action on a generic CPTP map Ψ^B is

$$\Theta^{B\to C}\left[\Psi^B\right]\left(\rho^{C_0}\right)=\operatorname{Tr}\left[\Psi^{B_0\to C_0}\left(u^{B_0}\right)Y^{C_0}\left(\rho^{C_0}\right)^{\mathrm{T}}Y^{C_0}\right]u^{C_1}, \tag{4}$$

where all systems are qubits, u is the maximally mixed state, and Y is the Pauli Y matrix (ρ^{C_0} is a generic density matrix, used to define the action of the CPTNI map $\Theta^{B\to C}\left[\Psi^B\right]$ on its input). Full details are presented in Appendix C.

We find that the right condition to ensure that a CPTNI-preserving supermap $\Theta^{B\to C}$ is physical is that it be *completely* CPTNI-preserving. This means that it is CPTNI-preserving even when tensored with the identity supermap:

$$\operatorname{Tr}\left[\left(\mathscr{I}^A\otimes\Theta^{B\to C}\right)\left[\Psi^{AB}\right]\left(\rho^{A_0C_0}\right)\right]\leq 1, \tag{5}$$

where Ψ^{AB} is a CPTP map, and $\rho^{A_0C_0}\in\mathscr{D}\left(\mathscr{H}^{A_0C_0}\right)$. The example in Eq. (4) highlights that, in general, not all CPTNI-preserving supermaps are completely CPTNI-preserving.

For superchannels the situation is different: it is sufficient to demand that they be CPP and TP-preserving (TPP), without requiring that they be TPP in a *complete* sense [31]. The situation of generic supermaps is also different from linear maps acting on quantum states. In the latter case, to have a physical CP map, it is enough to require that it be TNI, without demanding it in a complete sense. The ultimate reason for these different behaviors is related to causality and no-signaling [15], and it is fully examined in Appendix E.

Following [14, 31], we work in the Choi picture for quantum maps and supermaps. A summary of useful facts is presented in Appendix B.1. Let $\mathbf{J}_\Theta^{BC}$ be the Choi matrix of a supermap $\Theta^{B\to C}$, and J_Ψ^B the Choi matrix of a linear map $\Psi^B\in\mathscr{L}^B$. Then $\Theta^{B\to C}$ is a CPTNI-preserving supermap if and only if $\mathbf{J}_\Theta^{BC}\geq 0$ (since it is CPP), and it satisfies the additional condition deriving from Eq. (2):

$$\operatorname{Tr}\left[\mathbf{J}_\Theta^{BC_0}\left(J_\Psi^B\otimes\rho^{C_0}\right)^{\mathrm{T}}\right]\leq 1, \tag{6}$$

for every CPTP map $\Psi^B\in\mathscr{L}^B$, and every $\rho^{C_0}\in\mathscr{D}\left(\mathscr{H}^{C_0}\right)$ (see Appendix B.1). In a similar spirit, we can express the requirement of complete CPTNI preservation in Eq. (5) in the Choi picture as $\mathbf{J}_\Theta^{BC}\geq 0$ plus the remarkably simple additional constraint

$$\operatorname{Tr}\left[\mathbf{J}_\Theta^{BC_0}\left(M^{BC_0}\right)^{\mathrm{T}}\right]\leq 1, \tag{7}$$

for every positive semi-definite matrix M^{BC_0} with marginal $M^{B_0C_0}=I^{B_0}\otimes\rho^{C_0}$, for some $\rho^{C_0}\in\mathscr{D}\left(\mathscr{H}^{C_0}\right)$. The technical details are provided in Appendix B.2.

It is not hard to check that all the matrices of the form $J_\Psi^B\otimes\rho^{C_0}$ are a strict subset of the matrices M^{BC_0}, confirming that complete CPTNI preservation is at least as strict a condition as CPTNI preservation. In fact, it is stricter, as our counterexample in Eq. (4) shows: the supermap in Eq. (4) is CPTNI-preserving but *not completely* CPTNI-preserving. Consequently, the set of completely CPTNI-preserving supermaps is strictly contained in the set of CPTNI-preserving supermaps. The situation is illustrated in Fig. 1.

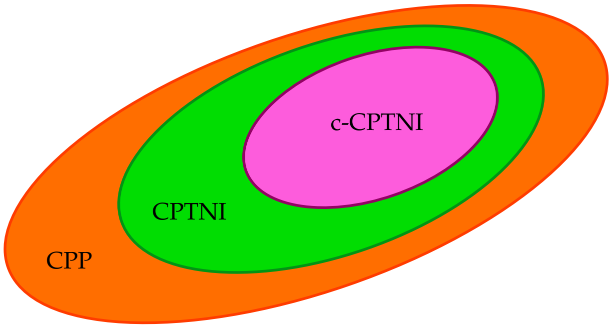

Figure 1: Inclusions between sets of supermaps. Here c-CPTNI denotes completely CPTNI-preserving supermaps.

To obtain our main result, namely the characterization of which CPTNI-preserving supermaps are physical, we consider a semi-definite program (SDP) inspired by Eq. (7):

$$\begin{aligned} \text{Find} \qquad & \alpha = \max_M \operatorname{Tr}\left[\mathbf{J}_\Theta^{BC_0}\left(M^{BC_0}\right)^{\mathrm{T}}\right] \\ \text{Subject to:} \qquad & M^{BC_0} \geq 0 \\ & M^{B_0C_0} = I^{B_0} \otimes \rho^{C_0}. \end{aligned} \tag{8}$$

If we consider the dual of the SDP (8)

$$\begin{aligned} \text{Find} \qquad & \beta = |B_0| \min r \\ \text{Subject to:} \qquad & r\,|B_0|\,\mathbf{J}_\Phi^{B_0C_0} \otimes u^{B_1} \geq \mathbf{J}_\Theta^{BC_0} \\ & \mathbf{J}_\Phi^{B_0C_0} \geq 0 \\ & \mathbf{J}_\Phi^{B_1C_0} = I^{B_1C_0} \\ & r \geq 0 \\ & r \in \mathbb{R} \\ & \Phi \text{ superchannel,} \end{aligned}$$

we convert Eq. (8) from an SDP having a constraint on M^{BC_0} into one having an explicit condition on $\mathbf{J}_\Theta^{BC_0}$. This condition is exactly what we need to derive our main result.

Theorem 1. *A CPTNI-preserving supermap can be completed to a superchannel if and only if it is* completely *CPTNI-preserving.*

The full proof is presented in Appendix D.

To summarize, for the first time we exactly pinned down the necessary and sufficient conditions governing the construction of quantum super-instruments. Specifically, we determined that only completely CPTNI-preserving supermaps can be implemented in a quantum super-instrument. Additionally, we showed an explicit example of a supermap that is CPTNI-preserving, but not completely CPTNI-preserving (Eq. (4)). Viewing CPTNI preservation as a higher-order generalization of the CPTNI condition for quantum maps, we cannot fail to note the difference between the theory of quantum supermaps—where CPTNI maps are regarded as states—and quantum theory. Indeed, in quantum theory, all CPTNI maps Ψ^B are also completely CPTNI, the latter meaning

$$\operatorname{Tr}\left[\left(\mathrm{id}^{A_0} \otimes \Psi^B\right)\left(\rho^{A_0B_0}\right)\right] \leq 1, \tag{9}$$

for every $\rho^{A_0B_0} \in \mathscr{D}\left(\mathscr{H}^{A_0B_0}\right)$. The ultimate reason for this difference is that the theory of quantum supermaps does not satisfy the fundamental property of causality [15].

Axiom 2 (Causality)**.** *The probability of a transformation occurring in an experiment is independent of the choice of experiments performed on its output.*

Loosely speaking, causality means that information cannot "come back from the future". One of its consequences is that all bipartite states are non-signaling. The existence of signaling bipartite channels [7, 39] is a clear signature that causality does not hold in the theory of quantum supermaps. A rigorous proof of this, and of the implications of the failure of causality for the theory of quantum supermaps, is presented in Appendix E.

The results we obtained in this letter improve our understanding of the operational viewpoint in quantum theory, and more generally in physics. In particular, we showed that the correct conditions to impose on a linear transformation to guarantee its physicality, be it a quantum map or a quantum supermap, must always be formulated in a *complete sense*. This means that they must always involve the tensor product with the identity transformation. Thus, for quantum supermaps we have the CPP condition and the complete CPTNI preservation condition. For quantum maps we have the CP condition and the complete TNI condition of Eq. (9). Since quantum theory satisfies causality, Eq. (9) becomes *equivalent* to the TNI condition we impose ordinarily on quantum maps (see Appendix E.2). However, the fundamental requirement is still the one expressed by Eq. (9).

The fact that conditions expressed in a complete sense are the right thing to demand is apparent if one adopts the framework of operational probabilistic theories [15, 16, 35, 36], presented in Appendix E. This is an operational approach to physical theories based on the notion of circuits, and of composition of physical transformations occurring in experiments. Our results confirm and strengthen the validity of this approach to the study of the fundamental operational properties of physical theories.

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A General facts about quantum maps and supermaps

Quantum maps describe the evolution of quantum systems, in both the deterministic and the probabilistic case (e.g. when a measurement is performed). To be consistent with the interpretation of mixed states as probabilistic ensembles, a quantum map Ψ^B must be linear, $\Psi^B \in \mathscr{L}^B$. This is not enough, because a quantum map must send quantum states to quantum states even when applied only to half of a bipartite state. For this reason we first demand that it be *completely positive* (CP): for every $\rho^{A_0B_0} \in \mathscr{D}\left(\mathscr{H}^{A_0B_0}\right)$ it must be $\left(\mathrm{id}^{A_0} \otimes \Psi^B\right)\left(\rho^{A_0B_0}\right) \geq 0$, where id^{A_0} is the identity map on system A_0. This means that Ψ^B sends positive semi-definite operators to positive semi-definite operators even when tensored with the identity. We also require that a map Ψ^B be *trace non-increasing* (TNI): $\mathrm{Tr}\left[\Psi^B\left(\rho^{B_0}\right)\right] \leq 1$, for every $\rho^{B_0} \in \mathscr{D}\left(\mathscr{H}^{B_0}\right)$.

In particular, if the trace is preserved, that is $\mathrm{Tr}\left[\Psi^B\left(\rho^{B_0}\right)\right] = 1$, for every $\rho^{B_0} \in \mathscr{D}\left(\mathscr{H}^{B_0}\right)$, we say that the map is *trace-preserving* (TP). The allowed quantum maps are those that are both CP and TNI (CPTNI). CPTP maps are also called *quantum channels* and represent the most general deterministic evolutions a quantum system can undergo. CPTNI maps that are not CPTP represent non-deterministic transformations. This is what happens in a quantum measurement, which can be seen as a collection of CPTNI maps $\left\{\Psi_x^B\right\}$, indexed by the outcomes x of that measurement, such that $\sum_x \Psi_x^B$ is a CPTP map. If we know the outcome x of the measurement, then we know that the system evolved under the CPTNI map Ψ_x^B. We can therefore construct a *quantum instrument*

$$\mathscr{E}^{B_0\to X_1B_1} = \sum_x |x\rangle\langle x|^{X_1} \otimes \Psi_x^B, \tag{10}$$

where $\left\{|x\rangle^{X_1}\right\}$ is an orthonormal basis of system X_1. $\mathscr{E}^{B_0\to XB_1}$ is a quantum channel with classical-quantum output. Here X_1 is the classical system, recording the measurement outcome. As such it represents the meter read by the experimenter performing the quantum measurement $\left\{\Psi_x^B\right\}$.

These notions can be easily generalized to quantum supermaps [13, 14, 47], namely to transformations sending quantum maps to quantum maps. Again these are linear maps, and an easy translation of the requirements of CP and TNI leads to the requirement of CPP (Eq. (1)) [13, 31] and TNI preservation. Specifically, a map is CPTNI-preserving if it is CPP, and sends CPTNI maps to CPTNI maps:

$$\mathrm{Tr}\left[\Theta^{B\to C}\left[\Psi^B\right]\left(\rho^{C_0}\right)\right] \leq 1, \tag{11}$$

for any CPTNI map Ψ^B and any $\rho^{C_0} \in \mathscr{D}\left(\mathscr{H}^{C_0}\right)$. In fact, if $\Theta^{B\to C}$ is CPP, it is enough to require that inequality (11) be satisfied by quantum channels Ψ^B, namely by CPTP maps.

To see it, let Ψ^B be a CPTNI map. We can always find another CPTNI map Ψ'^B such that $\Psi^B + \Psi'^B$ is CPTP. Now assume that $\Theta^{B\to C}$ is CPP, and sends CPTP maps to CPTNI maps. Then, for every $\rho^{B_0} \in \mathscr{D}\left(\mathscr{H}^{B_0}\right)$,

$$1 \geq \mathrm{Tr}\left[\Theta^{B\to C}\left[\Psi^B + \Psi'^B\right]\left(\rho^{B_0}\right)\right] = \mathrm{Tr}\left[\Theta^{B\to C}\left[\Psi^B\right]\left(\rho^{B_0}\right)\right] + \mathrm{Tr}\left[\Theta^{B\to C}\left[\Psi'^B\right]\left(\rho^{B_0}\right)\right].$$

Since $\Theta^{B\to C}$ is CPP, then $\mathrm{Tr}\left[\Theta^{B\to C}\left[\Psi^B\right]\left(\rho^{B_0}\right)\right] \geq 0$ and $\mathrm{Tr}\left[\Theta^{B\to C}\left[\Psi'^B\right]\left(\rho^{B_0}\right)\right] \geq 0$, therefore we conclude that it must be $\mathrm{Tr}\left[\Theta^{B\to C}\left[\Psi^B\right]\left(\rho^{B_0}\right)\right] \leq 1$, which means that $\Theta^{B\to C}$ satisfies Eq. (11).

A CPTNI-preserving supermap $\Theta^{B\to C}$ is called *superchannel* if it sends CPTP maps to CPTP maps [31]. The original definition in [18] required that it should send quantum channels to quantum channels in a complete sense, i.e. even when tensored with the identity supermap. In other words,

$$\mathrm{Tr}\left[\left(\mathscr{I}^A \otimes \Theta^{B\to C}\right)\left[\Psi^{AB}\right]\left(\rho^{A_0C_0}\right)\right] = 1.$$

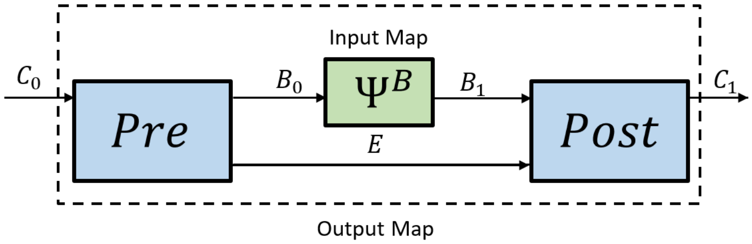


Figure 2: Realization of a superchannel. Here an input quantum channel is inserted between a CPTP pre-processing map and a CPTP post-processing map. The output is another quantum channel. The ancillary system E plays the role of a quantum memory between the pre- and the post-processing.

for any CPTP map Ψ^{AB} and any $\rho^{A_0C_0} \in \mathscr{D}\left(\mathscr{H}^{A_0C_0}\right)$, where $\mathscr{I}^A$ is the identity supermap on A. Actually, in [31, theorem 1], using the Choi picture, it was proved that we need not consider this requirement in a complete sense: a CPTNI-preserving supermap $\Theta^{B\to C}$ is a superchannel if and only if

$$\mathrm{Tr}\left[\Theta^{B\to C}\left[\Psi^B\right]\left(\rho^{C_0}\right)\right] = 1,$$

for any CPTP map Ψ^B and any $\rho^{C_0} \in \mathscr{D}\left(\mathscr{H}^{C_0}\right)$. In Appendix E.2 we will prove this result in an alternative way, without using the Choi isomorphism.

Superchannels are intimately related to channels: it was proven that all superchannels can be represented in terms of a pre- and a post-processing CPTP map [13, 31], as depicted in Fig. 2. Such a representation is called a quantum 1-comb [13]. From this we can get an intuitive grasp of the failure of causality in the theory of quantum supermaps. Consider a superchannel $\Theta^{B\to C}$ acting on a CPTNI map Ψ^B, which means that the superchannel occurs *after* Ψ^B is prepared in a laboratory or in a quantum circuit. The presence of a pre-processing in the realization of every $\Theta^{B\to C}$ implies that $\Theta^{B\to C}$ acts on the *input* of Ψ^B too, meaning, in some sense, that part of $\Theta^{B\to C}$ also acts *before* Ψ^B. Somehow in this situation there is *not* a well-defined notion of what comes "before" and "after", so causality cannot hold; for it would select a clear "arrow of time" in information processing. A more formal treatment and a proof of this fact is provided in Appendix E.

Superchannels play an important role because they represent all physical ways a quantum channel can evolve in an open system, and provide a framework for measurements on quantum operations, called *super-measurements*. These are described by a set $\left\{\Theta_x^{B\to C}\right\}$ of CPTNI-preserving supermaps such that $\sum_x \Theta_x^{B\to C}$ is a superchannel. Then we can construct a *quantum super-instrument* as the generalization of the quantum notion (see Eq. (3)):

$$\Upsilon^{B\to X_1C}\left[\Psi^B\right] = \sum_x |x\rangle\langle x|^{X_1} \otimes \Theta_x^{B\to C}\left[\Psi^B\right],$$

where system X_1 again represents the classical meter, like in Eq. (10). The main result of this letter is that, unlike CPTNI quantum maps, *not all* CPTNI-preserving supermaps can be part of a quantum super-instrument. In Appendix E we link this fact to the failure of causality [15] in the theory of quantum supermaps.

B The Choi picture for quantum maps and supermaps

In this Appendix we collect some results about the Choi representation of quantum maps and supermaps. Specifically, Appendix B.1 provides the general background information and some basic results about the Choi isomorphism. Appendix B.2 instead focuses on the derivations related to complete CPTNI preservation in the Choi form, and in particular on obtaining Eq. (7).

B.1 Choi matrices of quantum maps and supermaps

The first ingredient to define the Choi isomorphism is to consider the super-normalized maximally entangled state $|\phi_+\rangle^{A_0\widetilde{A}_0} = \sum_{j=1}^{|A_0|} |j\rangle^{A_0} |j\rangle^{\widetilde{A}_0}$, where $\left\{|j\rangle^{A_0}\right\}$ is a *fixed* orthonormal basis of $\mathscr{H}^{A_0}$ (and therefore of $\mathscr{H}^{\widetilde{A}_0}$ too, since $\widetilde{A}_0$ is just another copy of A_0). The Choi matrix of a linear map $\Psi^A \in \mathscr{L}^A$ is defined as$J_\Psi^A := \left(\mathrm{id}^{A_0} \otimes \Psi^{\widetilde{A}_0 \to A_1}\right)\left(\phi_+^{A_0\widetilde{A}_0}\right)$,where $\phi_+^{A_0\widetilde{A}_0} := |\phi_+\rangle\langle\phi_+|^{A_0\widetilde{A}_0}$. Again, since $\widetilde{A}_0$ is just another copy of A_0, the linear map $\Psi^{\widetilde{A}_0 \to A_1}$ is well defined.

In particular, Ψ^A is CP if and only if $J_\Psi^A \geq 0$. Ψ^A is CPTP if in addition one has $J_\Psi^{A_0} = I^{A_0}$. Instead, Ψ^A is CPTNI if, besides $J_\Psi^A \geq 0$, one has $J_\Psi^{A_0} \leq I^{A_0}$. The Choi matrix J_Ψ^A encodes all the information about Ψ^A because one can reconstruct the action of Ψ^A on quantum states from its Choi matrix:

$$\Psi^A\left(\rho^{A_0}\right) = \mathrm{Tr}_{A_0}\left[J_\Psi^A\left(\left(\rho^{A_0}\right)^{\mathrm{T}} \otimes I^{A_1}\right)\right], \tag{12}$$

for every $\rho^{A_0} \in \mathscr{D}\left(\mathscr{H}^{A_0}\right)$.

To define the Choi matrix of a supermap $\Theta^{A\to B}$, we follow the approach presented in [31]. Let us consider the following basis of the space $\mathscr{L}^A$:

$$\mathscr{E}_{jklm}^A\left(\rho^{A_0}\right) = \langle j|\rho|k\rangle^{A_0} |l\rangle\langle m|^{A_1},$$

for $j,k \in \{1,\ldots,|A_0|\}$ and $l,m \in \{1,\ldots,|A_1|\}$. The Choi matrix of the supermap $\Theta^{A\to B}$ can be defined as

$$\mathbf{J}_\Theta^{AB} := \sum_{j,k,l,m} J_{\mathscr{E}_{jklm}}^A \otimes J_{\Theta\left[\mathscr{E}_{jklm}\right]}^B.$$

Again, $\mathbf{J}_\Theta^{AB}$ encodes all the information about $\Theta^{A\to B}$. For instance, $\Theta^{A\to B}$ is CPP if and only if $\mathbf{J}_\Theta^{AB} \geq 0$. Moreover, we can express the action of a supermap on a quantum map Ψ^A using their Choi matrices: if $\Phi^B = \Theta^{A\to B}\left[\Psi^A\right]$, we have [31]

$$J_\Phi^B = \mathrm{Tr}_A\left[\mathbf{J}_\Theta^{AB}\left(\left(J_\Psi^A\right)^{\mathrm{T}} \otimes I^B\right)\right]. \tag{13}$$

A full characterization of superchannels from their Choi matrices was given in [31]: $\Theta^{A\to B}$ is a superchannel if and only if $\mathbf{J}_\Theta^{AB} \geq 0$, and one has $\mathbf{J}_\Theta^{AB_0} = \mathbf{J}_\Theta^{A_0B_0} \otimes u^{A_1}$ and $\mathbf{J}_\Theta^{A_1B_0} = I^{A_1B_0}$. Here $u^{A_1} = \frac{1}{|A_1|} I^{A_1}$ is the maximally mixed state. The fact that superchannels send CPTP maps to CPTP maps can equivalently be written as $\mathrm{Tr}\left[\Theta^{B\to C}\left[\Psi^B\right]\left(\rho^{C_0}\right)\right] = 1$, for every CPTP map Ψ^B and every density matrix ρ^{C_0}. Combining Eqs. (12) and (13), we have

$$\Theta^{B\to C}\left[\Psi^B\right]\left(\rho^{C_0}\right) = \mathrm{Tr}_{BC_0}\left[\mathbf{J}_\Theta^{BC}\left(\left(J_\Psi^B \otimes \rho^{C_0}\right)^{\mathrm{T}} \otimes I^{C_1}\right)\right];$$

therefore,

$$\mathrm{Tr}\left[\Theta^{B\to C}\left[\Psi^B\right]\left(\rho^{C_0}\right)\right] = \mathrm{Tr}_{C_1}\left\{\mathrm{Tr}_{BC_0}\left[\mathbf{J}_\Theta^{BC}\left(\left(J_\Psi^B \otimes \rho^{C_0}\right)^{\mathrm{T}} \otimes I^{C_1}\right)\right]\right\} = \mathrm{Tr}\left[\mathbf{J}_\Theta^{BC_0}\left(J_\Psi^B \otimes \rho^{C_0}\right)^{\mathrm{T}}\right]. \tag{14}$$

Hence $\Theta^{B\to C}$ is a superchannel if and only if

$$\operatorname{Tr}\left[\mathbf{J}_{\Theta}^{BC_0}\left(J_{\Psi}^{B}\otimes\rho^{C_0}\right)^{\mathrm{T}}\right]=1.$$

Similarly, we can characterize CPTNI-preserving supermaps in the Choi picture. By Eq. (2), a supermap is CPTNI-preserving if $\operatorname{Tr}\left[\Theta^{B\to C}\left[\Psi^{B}\right]\left(\rho^{C_0}\right)\right]\leq 1$, for every CPTP map Ψ^{B} and every density matrix ρ^{C_0}. By Eq. (14) we can rewrite this condition in the Choi picture as

$$\operatorname{Tr}\left[\mathbf{J}_{\Theta}^{BC_0}\left(J_{\Psi}^{B}\otimes\rho^{C_0}\right)^{\mathrm{T}}\right]\leq 1.$$

This proves Eq. (6).

B.2 Some technical derivations about completely CPTNI-preserving supermaps

Now we will focus on expressing the complete CPTNI preservation condition in the Choi picture. Looking at Eq. (5) tells us that we need to find an expression for $\operatorname{Tr}\left[\left(\mathscr{I}^{A}\otimes\Theta^{B\to C}\right)\left[\Psi^{AB}\right]\left(\rho^{A_0C_0}\right)\right]$ in the Choi picture. Note that the identity supermap does not change the systems it acts on. Therefore, to express Eq. (5) in the Choi form, we only consider how Ψ^{AB} is acted on by the supermap $\Theta^{B\to C}$, representing the action of the identity superchannel with the identity matrix I^{A}. Therefore combining Eqs. (12) and (13) this time yields

$$\begin{aligned}
\operatorname{Tr}\left[\left(\mathscr{I}^{A}\otimes\Theta^{B\to C}\right)\left[\Psi^{AB}\right]\left(\rho^{A_0C_0}\right)\right] &= \operatorname{Tr}\left[\left(I^{A}\otimes\mathbf{J}_{\Theta}^{BC}\right)\left(\left(J_{\Psi}^{AB}\right)^{\mathrm{T}_B}\otimes I^{C}\right)\left(\left(\rho^{A_0C_0}\right)^{\mathrm{T}}\otimes I^{A_1BC_1}\right)\right]= \\
&= \operatorname{Tr}_{A_0BC_0}\left\{\operatorname{Tr}_{A_1C_1}\left[\left(I^{A}\otimes\mathbf{J}_{\Theta}^{BC}\right)\left(\left(J_{\Psi}^{AB}\right)^{\mathrm{T}_B}\otimes I^{C}\right)\left(\left(\rho^{A_0C_0}\right)^{\mathrm{T}}\otimes I^{A_1BC_1}\right)\right]\right\}= \\
&= \operatorname{Tr}_{A_0BC_0}\left[\left(I^{A_0}\otimes\mathbf{J}_{\Theta}^{BC_0}\right)\left(\left(J_{\Psi}^{A_0B}\right)^{\mathrm{T}_B}\otimes I^{C_0}\right)\left(\left(\rho^{A_0C_0}\right)^{\mathrm{T}}\otimes I^{B}\right)\right].
\end{aligned}\tag{15}$$

Now let us define

$$M^{BC_0}:=\operatorname{Tr}_{A_0}\left[\left(\rho^{A_0C_0}\otimes I^{B}\right)\left(\left(J_{\Psi}^{A_0B}\right)^{\mathrm{T}_{A_0}}\otimes I^{C_0}\right)\right],\tag{16}$$

and let us calculate

$$\begin{aligned}
\operatorname{Tr}\left[\mathbf{J}_{\Theta}^{BC_0}\left(M^{BC_0}\right)^{\mathrm{T}}\right] &= \operatorname{Tr}_{A_0BC_0}\left[\left(I^{A_0}\otimes\mathbf{J}_{\Theta}^{BC_0}\right)\left(\left(J_{\Psi}^{A_0B}\right)^{\mathrm{T}_{A_0}}\otimes I^{C_0}\right)^{\mathrm{T}}\left(\rho^{A_0C_0}\otimes I^{B}\right)^{\mathrm{T}}\right] \\
&= \operatorname{Tr}\left[\left(I^{A_0}\otimes\mathbf{J}_{\Theta}^{BC_0}\right)\left(\left(J_{\Psi}^{A_0B}\right)^{\mathrm{T}_B}\otimes I^{C_0}\right)\left(\left(\rho^{A_0C_0}\right)^{\mathrm{T}}\otimes I^{B}\right)\right].
\end{aligned}$$

As we can see, this coincides with Eq. (15). Therefore $\operatorname{Tr}\left[\left(\mathscr{I}^{A}\otimes\Theta^{B\to C}\right)\left[\Psi^{AB}\right]\left(\rho^{A_0C_0}\right)\right]=\operatorname{Tr}\left[\mathbf{J}_{\Theta}^{BC_0}\left(M^{BC_0}\right)^{\mathrm{T}}\right]$, where M^{BC_0} is defined in Eq. (16). Now the complete CPTNI preservation condition of Eq. (5) becomes

$$\operatorname{Tr}\left[\mathbf{J}_{\Theta}^{BC_0}\left(M^{BC_0}\right)^{\mathrm{T}}\right]\leq 1\tag{17}$$

for every M^{BC_0} of the form (16). Note that $\operatorname{Tr}\left[\mathbf{J}_{\Theta}^{BC_0}\left(M^{BC_0}\right)^{\mathrm{T}}\right]\geq 0$ for every CPP supermap $\Theta^{B\to C}$, whence M^{BC_0} is positive semi-definite. Furthermore,

$$M^{B_0C_0}=\operatorname{Tr}_{A_0B_1}\left[\left(\rho^{A_0C_0}\otimes I^{B}\right)\left(\left(J_{\Psi}^{A_0B}\right)^{\mathrm{T}_{A_0}}\otimes I^{C_0}\right)\right]=\operatorname{Tr}_{A_0}\left[\rho^{A_0C_0}\right]\otimes I^{B_0}=I^{B_0}\otimes\rho^{C_0},$$

where we have used the fact that $\mathrm{Tr}_{A_0B_1}\left[\left(J_\Psi^{A_0B}\right)^{\mathrm{T}_{A_0}}\right]=\mathrm{Tr}_{A_0B_1}\left[J_\Psi^{A_0B}\right]$, and that $\mathrm{Tr}_{B_1}\left[J_\Psi^{A_0B}\right]=\mathrm{Tr}_{A_1B_1}\left[J_\Psi^{AB}\right]=I^{A_0}\otimes I^{B_0}$ because Ψ is CPTP (cf. Appendix B.1). So M^{BC_0} has marginal $M^{B_0C_0}=I^{B_0}\otimes\rho^{C_0}$.

Now we prove a key result, namely that *every* positive semi-definite matrix M^{BC_0} with marginal $M^{B_0C_0}=I^{B_0}\otimes\rho^{C_0}$, where ρ^{C_0} is any density matrix, can be written as in Eq. (16). In this way, instead of stating complete CPTNI preservation as in Eq. (17) for M^{BC_0} of the form (16), we will state it in a remarkably simpler way: $\Theta^{B\to C}$ is completely CPTNI-preserving if and only if Eq. (17) is satisfied for *any* positive semi-definite M^{BC_0} with marginal $M^{B_0C_0}=I^{B_0}\otimes\rho^{C_0}$. This technical result will be crucial for the main finding of this letter, namely the characterization of physical supermaps (see Appendix (D)).

Lemma 3. *Let $M^{BC_0}\geq 0$ such that $M^{B_0C_0}=I^{B_0}\otimes\rho^{C_0}$, for some $\rho^{C_0}\in\mathscr{D}\left(\mathscr{H}^{C_0}\right)$. Then*

$$M^{BC_0}=\mathrm{Tr}_{A_0}\left[\left(\rho^{A_0C_0}\otimes I^B\right)\left(\left(J_\Psi^{A_0B}\right)^{\mathrm{T}_{A_0}}\otimes I^{C_0}\right)\right],$$

where Ψ^{AB} is a CPTP map and $\rho^{A_0C_0}\in\mathscr{D}\left(\mathscr{H}^{A_0C_0}\right)$.

Proof. Let $\phi_+^{B_0\widetilde{B}_0}\otimes\varphi^{E_0C_0}$ be a purification of $M^{B_0C_0}=I^{B_0}\otimes\rho^{C_0}$, where $\varphi^{E_0C_0}\in\mathscr{D}\left(\mathscr{H}^{C_0E_0}\right)$ is a purification of ρ^{C_0}. Now let $\tau^{BC_0F_0}$ be a purification of M^{BC_0}, so $\tau^{BC_0F_0}$ is also a purification of $M^{B_0C_0}$. Thus, these two purifications can be related by an isometry channel $\mathscr{V}^{\widetilde{B}_0E_0\to B_1F_0}$ such that [33]

$$\tau^{BC_0F_0}=\left(\mathrm{id}^{B_0C_0}\otimes\mathscr{V}^{\widetilde{B}_0E_0\to B_1F_0}\right)\left(\phi_+^{B_0\widetilde{B}_0}\otimes\varphi^{E_0C_0}\right).$$

Performing the partial trace on system F_0 yields

$$M^{BC_0}=\left(\mathrm{id}^{B_0C_0}\otimes\Gamma^{\widetilde{B}_0E_0\to B_1}\right)\left(\phi_+^{B_0\widetilde{B}_0}\otimes\varphi^{E_0C_0}\right)\tag{18}$$

where $\Gamma^{\widetilde{B}_0E_0\to B_1}:=\mathrm{Tr}_{F_0}\circ\mathscr{V}^{\widetilde{B}_0E_0\to B_1F_0}$ is a CPTP map. The action of $\Gamma^{\widetilde{B}_0E_0\to B_1}$ on a generic state $\chi^{\widetilde{B}_0E_0}\in\mathscr{D}\left(\mathscr{H}^{\widetilde{B}_0E_0}\right)$ can be written in terms of its Choi matrix as

$$\Gamma^{\widetilde{B}_0E_0\to B_1}\left(\chi^{\widetilde{B}_0E_0}\right)=\mathrm{Tr}_{\widetilde{B}_0E_0}\left[J_\Gamma^{\widetilde{B}_0E_0B_1}\left(\left(\chi^{\widetilde{B}_0E_0}\right)^{\mathrm{T}}\otimes I^{B_1}\right)\right].\tag{19}$$

Let us substitute Eq. (19) into Eq. (18). Note that the identity map does not change the systems it acts on. Therefore, to express Eq. (18) in the Choi form, we only consider how $\phi_+^{B_0\widetilde{B}_0}\otimes\varphi^{E_0C_0}$ is acted on by the map $\Gamma^{\widetilde{B}_0E_0\to B_1}$, representing the action of the identity channel with the identity matrix $I^{B_0C_0}$. Thus Eq. (18) becomes

$$M^{BC_0}=\mathrm{Tr}_{\widetilde{B}_0E_0}\left[\left(I^{B_0C_0}\otimes J_\Gamma^{\widetilde{B}_0E_0B_1}\right)\left(\left(\phi_+^{B_0\widetilde{B}_0}\right)^{\mathrm{T}_{\widetilde{B}_0}}\otimes\left(\varphi^{E_0C_0}\right)^{\mathrm{T}_{E_0}}\otimes I^{B_1}\right)\right].$$

Expanding $\phi_+^{B_0\widetilde{B}_0}$, and using the cyclic property of the trace, we get

$$\mathrm{Tr}_{\widetilde{B}_0E_0}\left[\left(I^{B_0C_0}\otimes J_\Gamma^{E_0\widetilde{B}_0B_1}\right)\left(\left(\phi_+^{B_0\widetilde{B}_0}\right)^{\mathrm{T}_{\widetilde{B}_0}}\otimes\left(\varphi^{E_0C_0}\right)^{\mathrm{T}_{E_0}}\otimes I^{B_1}\right)\right]=$$

$$=\sum_{x,y}|x\rangle\langle y|^{B_0}\otimes\mathrm{Tr}_{E_0}\left[\left(I^{C_0}\otimes\left\langle x\middle|J_\Gamma^{E_0\widetilde{B}_0B_1}\middle|y\right\rangle^{\widetilde{B}_0}\right)\left(\left(\varphi^{E_0C_0}\right)^{\mathrm{T}_{E_0}}\otimes I^{B}\right)\right].$$

Since $\widetilde{B}_0$ is a copy of B_0,

$$\sum_{x,y} |x\rangle\langle y|^{B_0} \left\langle x \middle| J_\Gamma^{E_0\widetilde{B}_0 B_1} \middle| y \right\rangle^{\widetilde{B}_0} =: J_\Gamma^{E_0 B}$$

where we have replaced system $\widetilde{B}_0$ with system B_0, and we have set $B := B_0 B_1$ as usual. Now Γ is regarded as a channel from $B_0 E_0$ to B_1. With this in mind, we can write

$$M^{BC_0} = \mathrm{Tr}_{E_0}\left[\left(J_\Gamma^{E_0 B} \otimes I^{C_0}\right)\left(\left(\varphi^{E_0 C_0}\right)^{\mathrm{T}_{E_0}} \otimes I^B\right)\right].$$

Taking the transpose on E_0, this expression can be rewritten as

$$M^{BC_0} = \mathrm{Tr}_{E_0}\left[\left(\varphi^{E_0 C_0} \otimes I^B\right)\left(\left(J_\Gamma^{E_0 B}\right)^{\mathrm{T}_{E_0}} \otimes I^{C_0}\right)\right].$$

Now rename E_0 as A_0, and define $J_\Psi^{A_0 B} := J_\Gamma^{A_0 B}$, and $\rho^{A_0 C_0} := \varphi^{A_0 C_0}$. We find that M^{BC_0} can be written in the form of Eq. (16). □

This means that once we require M^{BC_0} to be positive semi-definite and with marginal $M^{B_0 C_0} = I^{B_0} \otimes \rho^{C_0}$, for some density matrix ρ^{C_0}, this automatically implies that M^{BC_0} has the special form of Eq. (16). Consequently, we can express the requirement of complete CPTNI preservation in the Choi form as follows: $\Theta^{B\to C}$ is complete CPTNI-preserving if and only if $\mathbf{J}_\Theta^{BC} \geq 0$ and $\mathrm{Tr}\left[\mathbf{J}_\Theta^{BC}\left(M^{BC_0}\right)^{\mathrm{T}}\right] \leq 1$ for every positive semi-definite M^{BC_0} with marginal $M^{B_0 C_0} = I^{B_0} \otimes \rho^{C_0}$, where $\rho^{C_0} \in \mathscr{D}\left(\mathscr{H}^{C_0}\right)$. This is Eq. (7). In particular, $\Theta^{B\to C}$ is a superchannel, which is a completely CPTP-preserving supermap if and only if $\mathrm{Tr}\left[\mathbf{J}_\Theta^{BC}\left(M^{BC_0}\right)^{\mathrm{T}}\right] = 1$ for every $M^{BC_0} \geq 0$ with $M^{B_0 C_0} = I^{B_0} \otimes \rho^{C_0}$, for $\rho^{C_0} \in \mathscr{D}\left(\mathscr{H}^{C_0}\right)$.

Note that, among these M^{BC_0}'s we can find matrices of the form $J_\Psi^B \otimes \rho^{C_0}$, where Ψ^B is a CPTP map. These matrices are those used to check the CPTNI preservation condition (cf. Eq. (6)). Indeed, $J_\Psi^B \otimes \rho^{C_0} \geq 0$, and the marginal is

$$\mathrm{Tr}_{B_1}\left[J_\Psi^B \otimes \rho^{C_0}\right] = \mathrm{Tr}_{B_1}\left[J_\Psi^B\right] \otimes \rho^{C_0} = I^{B_0} \otimes \rho^{C_0}$$

because J_Ψ^B is the Choi matrix of a CPTP map (see Appendix A). Therefore, as it must be, we recover in the Choi picture that CPTNI preservation is not stronger than complete CPTNI preservation. In fact, it is strictly weaker, as shown in the next Appendix C.

C A supermap that is CPTNI-preserving, but *not* completely CPTNI-preserving

In this Appendix we present the concrete counterexample of a supermap $\Theta^{B\to C}$ that is CPTNI-preserving, but *not completely* CPTNI-preserving. In this construction we take $|B_0| = |B_1| = |C_0| = 2$. Consider a supermap $\Theta^{B\to C}$ that has a Choi matrix with marginal $\mathbf{J}_\Theta^{BC_0} = I^{B_0} \otimes \psi_-^{B_1 C_0}$, where $\psi_-^{B_1 C_0} = |\psi_-\rangle\langle\psi_-|^{B_1 C_0}$, and $|\psi_-\rangle^{B_1 C_0} = \frac{1}{\sqrt{2}}\left(|01\rangle^{B_1 C_0} - |10\rangle^{B_1 C_0}\right)$ is the singlet state. Given this marginal, a possible Choi matrix of the supermap $\Theta^{B\to C}$ is $\mathbf{J}_\Theta^{BC} = I^{B_0} \otimes \psi_-^{B_1 C_0} \otimes u^{C_1}$, where u^{C_1} is the maximally mixed state of C_1. Now we will prove that this supermap is CPTNI-preserving, but *not completely* CPTNI-preserving.

To this end, we first show that $\mathbf{J}_\Theta^{BC}$ satisfies Eq. (6). If Ψ^B is a CPTP map and ρ^{C_0} is a density matrix, we have

$$\mathrm{Tr}\left[\mathbf{J}_\Theta^{BC_0}\left(J_\Psi^B \otimes \rho^{C_0}\right)^{\mathrm{T}}\right] = \mathrm{Tr}\left[\left(I^{B_0} \otimes \psi_-^{B_1 C_0}\right)\left(J_\Psi^B \otimes \rho^{C_0}\right)^{\mathrm{T}}\right].$$

Now we express $\psi_-^{B_1C_0}$ in terms of the super-normalized maximally entangled state $\phi_+^{B_1C_0}$:

$$\psi_-^{B_1C_0} = \frac{1}{2}\left(I^{B_1}\otimes Y^{C_0}\right)\phi_+^{B_1C_0}\left(I^{B_1}\otimes Y^{C_0}\right), \tag{20}$$

where Y^{C_0} is the Pauli Y matrix. Then

$$\begin{aligned}\operatorname{Tr}\left[\mathbf{J}_\Theta^{BC_0}\left(J_\Psi^B\otimes\rho^{C_0}\right)^{\mathrm{T}}\right] &= \frac{1}{2}\operatorname{Tr}\left[\left(I^{B_0}\otimes\left(I^{B_1}\otimes Y^{C_0}\right)\phi_+^{B_1C_0}\left(I^{B_1}\otimes Y^{C_0}\right)\right)\left(J_\Psi^B\otimes\rho^{C_0}\right)^{\mathrm{T}}\right] = \\ &= \frac{1}{2}\operatorname{Tr}_{B_1C_0}\left\{\operatorname{Tr}_{B_0}\left[\left(I^{B_0}\otimes\left(I^{B_1}\otimes Y^{C_0}\right)\phi_+^{B_1C_0}\left(I^{B_1}\otimes Y^{C_0}\right)\right)\left(J_\Psi^B\otimes\rho^{C_0}\right)^{\mathrm{T}}\right]\right\} = \\ &= \frac{1}{2}\operatorname{Tr}_{B_1C_0}\left[\phi_+^{B_1C_0}\left(\left(J_\Psi^{B_1}\right)^{\mathrm{T}}\otimes Y^{C_0}\left(\rho^{C_0}\right)^{\mathrm{T}}Y^{C_0}\right)\right],\end{aligned}$$

using the cyclic property of the trace. Now let us expand $\phi_+^{B_1C_0}$.

$$\begin{aligned}\frac{1}{2}\operatorname{Tr}\left[\phi_+^{B_1C_0}\left(\left(J_\Psi^{B_1}\right)^{\mathrm{T}}\otimes Y^{C_0}\left(\rho^{C_0}\right)^{\mathrm{T}}Y^{C_0}\right)\right] &= \frac{1}{2}\operatorname{Tr}\left[\sum_{x,y=1}^{2}\left|xx\right\rangle\left\langle yy\right|^{B_1C_0}\left(\left(J_\Psi^{B_1}\right)^{\mathrm{T}}\otimes Y^{C_0}\left(\rho^{C_0}\right)^{\mathrm{T}}Y^{C_0}\right)\right] = \\ &= \frac{1}{2}\operatorname{Tr}\left[\sum_{x,y=1}^{2}\left\langle y\middle|\left(J_\Psi^{B_1}\right)^{\mathrm{T}}\middle|x\right\rangle^{B_1}\left|x\right\rangle\left\langle y\right|^{C_0}Y^{C_0}\left(\rho^{C_0}\right)^{\mathrm{T}}Y^{C_0}\right] = \\ &= \frac{1}{2}\operatorname{Tr}\left[\sum_{x,y=1}^{2}\left\langle x\middle|J_\Psi^{B_1}\middle|y\right\rangle^{B_1}\left|x\right\rangle\left\langle y\right|^{C_0}Y^{C_0}\left(\rho^{C_0}\right)^{\mathrm{T}}Y^{C_0}\right].\end{aligned} \tag{21}$$

Here the expression $\sum_{x,y=1}^{2}\left\langle x\middle|J_\Psi^{B_1}\middle|y\right\rangle^{B_1}\left|x\right\rangle\left\langle y\right|^{C_0}$ means considering Ψ^B with its output system transformed from B_1 to C_0. With this simplification, Eq. (21) reads

$$\operatorname{Tr}\left[\mathbf{J}_\Theta^{BC_0}\left(J_\Psi^B\otimes\rho^{C_0}\right)^{\mathrm{T}}\right] = \frac{1}{2}\operatorname{Tr}\left[J_\Psi^{C_0}Y^{C_0}\left(\rho^{C_0}\right)^{\mathrm{T}}Y^{C_0}\right].$$

Now, both $\frac{1}{2}J_\Psi^{C_0}$ and $Y^{C_0}\left(\rho^{C_0}\right)^{\mathrm{T}}Y^{C_0}$ are density operators, therefore

$$\operatorname{Tr}\left[\mathbf{J}_\Theta^{BC_0}\left(J_\Psi^B\otimes\rho^{C_0}\right)^{\mathrm{T}}\right] = \operatorname{Tr}\left[\left(\frac{1}{2}J_\Psi^{C_0}\right)\left(Y^{C_0}\left(\rho^{C_0}\right)^{\mathrm{T}}Y^{C_0}\right)\right]\leq 1.$$

Hence $\mathbf{J}_\Theta^{BC_0}$ satisfies Eq. (6); therefore $\Theta^{B\to C}$ is a CPTNI-preserving supermap.

Now we show that $\Theta^{B\to C}$ violates Eq. (7). To this end, let us take $M^{BC_0} = \left(\mathbf{J}_\Theta^{BC_0}\right)^{\mathrm{T}}$. This choice of M^{BC_0} complies with the two requests on M^{BC_0} in Eq. (7). Since $\mathbf{J}_\Theta^{BC_0} = I^{B_0}\otimes\psi_-^{B_1C_0}$, $\left(\mathbf{J}_\Theta^{BC_0}\right)^{\mathrm{T}}$ is positive semi-definite; and its marginal

$$M^{B_0C_0} = \operatorname{Tr}_{B_1}\left[\left(\mathbf{J}_\Theta^{BC_0}\right)^{\mathrm{T}}\right] = \left(\operatorname{Tr}_{B_1}\left[\mathbf{J}_\Theta^{BC_0}\right]\right)^{\mathrm{T}} = \left[I^{B_0}\otimes u^{C_0}\right]^{\mathrm{T}} = I^{B_0}\otimes u^{C_0}$$

is of the form $I^{B_0}\otimes\rho^{C_0}$, with ρ^{C_0} density matrix. Then

$$\operatorname{Tr}\left[\mathbf{J}_\Theta^{BC_0}\left(M^{BC_0}\right)^{\mathrm{T}}\right] = \operatorname{Tr}\left[\left(I^{B_0}\otimes\psi_-^{B_1C_0}\right)\left(I^{B_0}\otimes\psi_-^{B_1C_0}\right)\right] = \operatorname{Tr}\left[I^{B_0}\otimes\left(\psi_-^{B_1C_0}\right)^2\right] =$$

$$= \mathrm{Tr}_{B_1C_0}\left\{\mathrm{Tr}_{B_0}\left[I^{B_0}\otimes\psi_-^{B_1C_0}\right]\right\} = 2\mathrm{Tr}\left[\psi_-^{B_1C_0}\right] = 2 > 1.$$

This is in contrast with Eq. (7), therefore the supermap $\Theta^{B\to C}$ *not* a *completely* CPTNI-preserving supermap, despite being CPTNI-preserving.

We conclude this Appendix by reconstructing $\Theta^{B\to C}$ from its Choi matrix $\mathbf{J}_\Theta^{BC} = I^{B_0}\otimes\psi_-^{B_1C_0}\otimes u^{C_1}$. By Eqs. (12) and (13), we have

$$\Theta^{B\to C}\left[\Psi^B\right]\left(\rho^{C_0}\right) = \mathrm{Tr}_{BC_0}\left[\mathbf{J}_\Theta^{BC}\left(J_\Psi^B\otimes\rho^{C_0}\otimes I^{C_1}\right)^{\mathrm{T}}\right] = \mathrm{Tr}_{BC_0}\left[\left(I^{B_0}\otimes\psi_-^{B_1C_0}\right)\left(J_\Psi^B\otimes\rho^{C_0}\right)^{\mathrm{T}}\right]u^{C_1} =$$

$$= \mathrm{Tr}_{B_1C_0}\left\{\mathrm{Tr}_{B_0}\left[\left(I^{B_0}\otimes\psi_-^{B_1C_0}\right)\left(J_\Psi^B\otimes\rho^{C_0}\right)^{\mathrm{T}}\right]\right\}u^{C_1} = \mathrm{Tr}\left[\psi_-^{B_1C_0}\left(J_\Psi^{B_1}\otimes\rho^{C_0}\right)^{\mathrm{T}}\right]u^{C_1}.$$

Recalling Eq. (20), we get

$$\Theta^{B\to C}\left[\Psi^B\right]\left(\rho^{C_0}\right) = \frac{1}{2}\mathrm{Tr}\left[\phi_+^{B_1C_0}\left(\left(J_\Psi^{B_1}\right)^{\mathrm{T}}\otimes Y^{C_0}\left(\rho^{C_0}\right)^{\mathrm{T}}Y^{C_0}\right)\right]u^{C_1},$$

and using an argument similar to the one in Eq. (21), we finally obtain

$$\Theta^{B\to C}\left[\Psi^B\right]\left(\rho^{C_0}\right) = \frac{1}{2}\mathrm{Tr}\left[J_\Psi^{C_0}Y^{C_0}\left(\rho^{C_0}\right)^{\mathrm{T}}Y^{C_0}\right]u^{C_1}. \tag{22}$$

By Eq. (12), $\Psi^{B_0\to C_0}\left(u^{B_0}\right) = \frac{1}{2}\mathrm{Tr}_{B_0}\left[J_\Psi^{B_0C_0}I^{B_0C_0}\right] = \frac{1}{2}J_\Psi^{C_0}$. An equivalent form of Eq. (22) is therefore

$$\Theta^{B\to C}\left[\Psi^B\right]\left(\rho^{C_0}\right) = \mathrm{Tr}\left[\Psi^{B_0\to C_0}\left(u^{B_0}\right)Y^{C_0}\left(\rho^{C_0}\right)^{\mathrm{T}}Y^{C_0}\right]u^{C_1}.$$

This is exactly Eq. (4).

D The main result

In this Appendix we prove the main result of this letter, namely that a supermap can be part of a superinstrument if and only if it is completely CPTNI-preserving. To this end, it is useful to consider the SDP (8), reported here for the reader's convenience.

$$\begin{array}{ll} \text{Find} & \alpha = \max\limits_M \mathrm{Tr}\left[\mathbf{J}_\Theta^{BC_0}\left(M^{BC_0}\right)^{\mathrm{T}}\right] \\ \text{Subject to:} & M^{BC_0}\geq 0 \\ & M^{B_0C_0} = I^{B_0}\otimes\rho^{C_0}. \end{array}$$

Theorem 4. *Suppose $\Theta^{B\to C}$ is a CPTNI-preserving supermap. Then there exists another CPTNI-preserving supermap $\Theta'^{B\to C}$ such that $\Theta^{B\to C}+\Theta'^{B\to C}$ is a superchannel if and only if $\Theta^{B\to C}$ is* completely *CPTNI-preserving.*

Proof. First we will show sufficiency, namely that any completely CPTNI-preserving supermap $\Theta^{B\to C}$ can be completed to a superchannel. Following [6], let us write the SDP (8) in a different form. To do so, consider the linear map $\mathscr{N}:\mathscr{B}_h\left(\mathscr{H}^{BC_0}\right)\to\mathbb{R}\oplus\mathscr{B}_h\left(\mathscr{H}^{B_0C_0}\right)$, defined as

$$\mathscr{N}\left(M^{BC_0}\right) := \left(\mathrm{Tr}\left[M^{BC_0}\right], M^{B_0C_0}-u^{B_0}\otimes M^{C_0}\right)$$

for every hermitian matrix M^{BC_0}. We are working with positive semi-definite matrices M^{BC_0} with marginal $M^{B_0C_0} = I^{B_0} \otimes \rho^{C_0}$, where $\rho^{C_0} \in \mathscr{D}\left(\mathscr{H}^{C_0}\right)$, whence

$$\operatorname{Tr}\left[M^{BC_0}\right] = \operatorname{Tr}_{B_0C_0}\left\{\operatorname{Tr}_{B_1}\left[M^{BC_0}\right]\right\} = \operatorname{Tr}\left[M^{B_0C_0}\right] = \operatorname{Tr}\left[I^{B_0} \otimes \rho^{C_0}\right] = |B_0|\,.$$

In addition,

$$M^{C_0} = \operatorname{Tr}_B\left[M^{BC_0}\right] = \operatorname{Tr}_{B_0}\left[M^{B_0C_0}\right] = \operatorname{Tr}_{B_0}\left[I^{B_0} \otimes \rho^{C_0}\right] = |B_0|\,\rho^{C_0}.$$

Using $\mathscr{N}$, we can replace the condition $M^{B_0C_0} = I^{B_0} \otimes \rho^{C_0}$ with $\mathscr{N}\left(M^{BC_0}\right) - \left(|B_0|\,, 0^{B_0C_0}\right) = \left(0, 0^{B_0C_0}\right)$. Rewriting the SDP (8) in terms of $\mathscr{N}$ one obtains

$$\begin{array}{rl} \text{Find} & \alpha = \max\limits_{M} \operatorname{Tr}\left[\mathbf{J}_{\Theta}^{BC_0}\left(M^{BC_0}\right)^{\mathrm{T}}\right] \\ \text{Subject to:} & \mathscr{N}\left(M^{BC_0}\right) - \left(|B_0|\,, 0^{B_0C_0}\right) = \left(0, 0^{B_0C_0}\right) \\ & M^{BC_0} \geq 0. \end{array}$$

We can now construct the associated dual problem as follows. The dual map of $\mathscr{N}$ is $\mathscr{N}^* : \mathbb{R} \oplus \mathscr{B}_h\left(\mathscr{H}^{B_0C_0}\right) \to \mathscr{B}_h\left(\mathscr{H}^{BC_0}\right)$ such that

$$\mathscr{N}^*\left(r, \sigma^{B_0C_0}\right) = \left(rI^{B_0C_0} + \sigma^{B_0C_0} - u^{B_0} \otimes \sigma^{C_0}\right) \otimes I^{B_1},$$

where $\left(r, \sigma^{B_0C_0}\right) \in \mathbb{R} \oplus \mathscr{B}_h\left(\mathscr{H}^{B_0C_0}\right)$. The dual problem is then

$$\begin{array}{rl} \text{Find} & \beta = \min\left\langle\left(r, \sigma^{B_0C_0}\right), \left(|B_0|\,, 0\right)\right\rangle \\ \text{Subject to:} & \left(rI^{B_0C_0} + \sigma^{B_0C_0} - u^{B_0} \otimes \sigma^{C_0}\right) \otimes I^{B_1} - \mathbf{J}_{\Theta}^{BC_0} \geq 0 \\ & r \in \mathbb{R} \\ & \sigma^{B_0C_0} \in \mathscr{B}_h\left(\mathscr{H}^{B_0C_0}\right), \end{array}$$

where the inner product $\left\langle\left(r, \sigma^{B_0C_0}\right), \left(s, \tau^{B_0C_0}\right)\right\rangle$ is defined as

$$\left\langle\left(r, \sigma^{B_0C_0}\right), \left(s, \tau^{B_0C_0}\right)\right\rangle = rs + \operatorname{Tr}\left[\sigma^{B_0C_0}\tau^{B_0C_0}\right].$$

With this in mind, the dual problem simplifies to

$$\begin{array}{rl} \text{Find} & \beta = |B_0| \min r \\ \text{Subject to:} & \left(rI^{B_0C_0} + \sigma^{B_0C_0} - u^{B_0} \otimes \sigma^{C_0}\right) \otimes I^{B_1} \geq \mathbf{J}_{\Theta}^{BC_0} \\ & r \in \mathbb{R} \\ & \sigma^{B_0C_0} \in \mathscr{B}_h\left(\mathscr{H}^{B_0C_0}\right). \end{array} \tag{23}$$

Notice that the matrix $rI^{B_0C_0} + \sigma^{B_0C_0} - u^{B_0} \otimes \sigma^{C_0}$ must be positive semi-definite, otherwise the first constraint could not be satisfied. In particular this implies $r \geq 0$. Indeed, if $r < 0$, for some $\sigma^{B_0C_0}$ the matrix $rI^{B_0C_0} + \sigma^{B_0C_0} - u^{B_0} \otimes \sigma^{C_0}$ would have negative eigenvalues. Factoring $r\,|B_0|$ out of the first term of the constraint in Eq. (23), we get

$$\left(rI^{B_0C_0} + \sigma^{B_0C_0} - u^{B_0} \otimes \sigma^{C_0}\right) \otimes I^{B_1} = r\,|B_0|\left(u^{B_0} \otimes I^{C_0} + \sigma'^{B_0C_0} - u^{B_0} \otimes \sigma'^{C_0}\right) \otimes I^{B_1},$$

where $\sigma'^{B_0C_0} := \frac{1}{r|B_0|}\sigma^{B_0C_0}$ if $r \neq 0$. Note that this does not alter the constraint on the dual SDP, so we can forget the primes, and rewrite Eq. (23) as

$$\begin{array}{rl} \text{Find} & \beta = |B_0| \min r \\ \text{Subject to:} & r\,|B_0|\left(u^{B_0}\otimes I^{C_0}+\sigma^{B_0C_0}-u^{B_0}\otimes\sigma^{C_0}\right)\otimes I^{B_1} \geq \mathbf{J}_{\Theta}^{BC_0} \\ & r \geq 0 \\ & \sigma^{B_0C_0} \in \mathscr{B}_h\left(\mathscr{H}^{B_0C_0}\right). \end{array}$$

In particular this implies that $u^{B_0}\otimes I^{C_0}+\sigma^{B_0C_0}-u^{B_0}\otimes\sigma^{C_0} \geq 0$. Now let us define

$$\mathbf{J}_{\Phi}^{BC_0} := \left(u^{B_0}\otimes I^{C_0}+\sigma^{B_0C_0}-u^{B_0}\otimes\sigma^{C_0}\right)\otimes I^{B_1}. \tag{24}$$

Note that $\mathbf{J}_{\Phi}^{BC_0} = \mathbf{J}_{\Phi}^{B_0C_0}\otimes u^{B_1}$ because

$$\mathbf{J}_{\Phi}^{B_0C_0} = \mathrm{Tr}_{B_1}\left[\mathbf{J}_{\Phi}^{BC_0}\right] = |B_1|\left(u^{B_0}\otimes I^{C_0}+\sigma^{B_0C_0}-u^{B_0}\otimes\sigma^{C_0}\right).$$

Moreover,

$$\mathbf{J}_{\Phi}^{B_1C_0} = \mathrm{Tr}_{B_0}\left[\mathbf{J}_{\Phi}^{BC_0}\right] = \left(I^{C_0}+\sigma^{C_0}-\sigma^{C_0}\right)\otimes I^{B_1} = I^{B_1C_0}.$$

Since $\mathbf{J}_{\Phi}^{B_0C_0} \geq 0$, by Appendix B.1 $\mathbf{J}_{\Phi}^{BC_0}$ is the marginal Choi matrix of a superchannel $\Phi^{B\to C}$. Eq. (24) can be taken as the definition of the marginal $\mathbf{J}_{\Phi}^{BC_0}$ of the Choi matrix of *any* superchannel. This is because any such marginal $\mathbf{J}_{\Phi}^{BC_0}$ can be written as in Eq. (24) for some hermitian matrix $\sigma^{B_0C_0}$: it is enough to take $\sigma^{B_0C_0}$ to be $\frac{1}{|B_1|}\mathbf{J}_{\Phi}^{B_0C_0}$. Indeed, substituting $\sigma^{B_0C_0} = \frac{1}{|B_1|}\mathbf{J}_{\Phi}^{B_0C_0}$ in the right-hand side of Eq. (24) yields

$$\begin{aligned} |B_1|\left(u^{B_0}\otimes I^{C_0}+\frac{1}{|B_1|}\mathbf{J}_{\Phi}^{B_0C_0}-\frac{1}{|B_1|}u^{B_0}\otimes\mathbf{J}_{\Phi}^{C_0}\right)\otimes u^{B_1} &= \left(|B_1|\,u^{B_0}\otimes I^{C_0}+\mathbf{J}_{\Phi}^{B_0C_0}-u^{B_0}\otimes\mathbf{J}_{\Phi}^{C_0}\right)\otimes u^{B_1} \\ &= \left(|B_1|\,u^{B_0}\otimes I^{C_0}+\mathbf{J}_{\Phi}^{B_0C_0}-u^{B_0}\otimes\mathrm{Tr}_{BC_1}\left[\mathbf{J}_{\Phi}^{BC}\right]\right)\otimes u^{B_1} \\ &= \left(|B_1|\,u^{B_0}\otimes I^{C_0}+\mathbf{J}_{\Phi}^{B_0C_0}-u^{B_0}\otimes\mathrm{Tr}_{B_1}\left\{\mathrm{Tr}_{B_0C_1}\left[\mathbf{J}_{\Phi}^{BC}\right]\right\}\right)\otimes u^{B_1} \\ &= \left(|B_1|\,u^{B_0}\otimes I^{C_0}+\mathbf{J}_{\Phi}^{B_0C_0}-u^{B_0}\otimes\mathrm{Tr}_{B_1}\left[\mathbf{J}_{\Phi}^{B_1C_0}\right]\right)\otimes u^{B_1} \\ &= \left(|B_1|\,u^{B_0}\otimes I^{C_0}+\mathbf{J}_{\Phi}^{B_0C_0}-u^{B_0}\otimes\mathrm{Tr}_{B_1}\left[I^{B_1C_0}\right]\right)\otimes u^{B_1} \\ &= \left(|B_1|\,u^{B_0}\otimes I^{C_0}+\mathbf{J}_{\Phi}^{B_0C_0}-|B_1|\,u^{B_0}\otimes I^{C_0}\right)\otimes u^{B_1} \\ &= \mathbf{J}_{\Phi}^{B_0C_0}\otimes u^{B_1}. \end{aligned}$$

Therefore, in the light of these remarks, the dual SDP can be equivalently formulated in the following terms:

$$\begin{array}{rl} \text{Find} & \beta = |B_0| \min r \\ \text{Subject to:} & r\,|B_0|\,\mathbf{J}_{\Phi}^{B_0C_0}\otimes u^{B_1} \geq \mathbf{J}_{\Theta}^{BC_0} \\ & \mathbf{J}_{\Phi}^{B_0C_0} \geq 0 \\ & \mathbf{J}_{\Phi}^{B_1C_0} = I^{B_1C_0} \\ & r \geq 0. \end{array}$$

Strong duality states that the primal and dual problem have the same optimal solution, therefore $\alpha = \beta$. Since $\Theta^{B\to C}$ is completely CPTNI-preserving, $\alpha = \max_M \operatorname{Tr}\left[\mathbf{J}_{\Theta}^{BC_0}\left(M^{BC_0}\right)^{\mathrm{T}}\right] \leq 1$. Hence $\beta \leq 1$. Clearly taking $r\left|B_0\right| = \beta$ satisfies the constraint $r\left|B_0\right|\mathbf{J}_{\Phi}^{B_0C_0}\otimes u^{B_1} \geq \mathbf{J}_{\Theta}^{BC_0}$, and we have

$$\mathbf{J}_{\Phi}^{B_0C_0}\otimes u^{B_1} \geq \beta\mathbf{J}_{\Phi}^{B_0C_0}\otimes u^{B_1} \geq \mathbf{J}_{\Theta}^{BC_0}$$

because $\beta \leq 1$. Now define $\Theta'^{B\to C}$ to be a new supermap such that $\mathbf{J}_{\Theta'}^{BC_0} := \mathbf{J}_{\Phi}^{B_0C_0}\otimes u^{B_1} - \mathbf{J}_{\Theta}^{BC_0}$. By construction $\mathbf{J}_{\Theta'}^{BC_0} \geq 0$; and by substituting $\mathbf{J}_{\Theta'}^{BC_0}$ into the left-hand side of Eq. (6) one obtains

$$\operatorname{Tr}\left[\mathbf{J}_{\Theta'}^{BC_0}\left(J_{\Psi}^{B}\otimes\rho^{C_0}\right)^{\mathrm{T}}\right] = \operatorname{Tr}\left[\mathbf{J}_{\Phi}^{BC_0} - \mathbf{J}_{\Theta}^{BC_0}\left(J_{\Psi}^{B}\otimes\rho^{C_0}\right)^{\mathrm{T}}\right] =$$

$$= \operatorname{Tr}\left[\mathbf{J}_{\Phi}^{BC_0}\left(J_{\Psi}^{B}\otimes\rho^{C_0}\right)^{\mathrm{T}}\right] - \operatorname{Tr}\left[\mathbf{J}_{\Theta}^{BC_0}\left(J_{\Psi}^{B}\otimes\rho^{C_0}\right)^{\mathrm{T}}\right] = 1 - \operatorname{Tr}\left[\mathbf{J}_{\Theta}^{BC_0}\left(J_{\Psi}^{B}\otimes\rho^{C_0}\right)^{\mathrm{T}}\right],$$

where we have used the fact that $\Phi^{B\to C}$ is a superchannel (see Appendix B.1). Now, $\operatorname{Tr}\left[\mathbf{J}_{\Theta}^{BC_0}\left(J_{\Psi}^{B}\otimes\rho^{C_0}\right)^{\mathrm{T}}\right] \geq 0$ because $\Theta^{B\to C}$ is CPP. Therefore $\operatorname{Tr}\left[\mathbf{J}_{\Theta'}^{BC_0}\left(J_{\Psi}^{B}\otimes\rho^{C_0}\right)^{\mathrm{T}}\right] \leq 1$ for every $J_{\Psi}^{B}\otimes\rho^{C_0}$, thus $\Theta'^{B\to C}$ is CPTNI-preserving.

To conclude the proof, let us prove necessity. Assume that $\Theta^{B\to C}$ is a CPTNI-preserving supermap such that $\Phi^{B\to C} = \Theta^{B\to C} + \Theta'^{B\to C}$ is a superchannel, where $\Theta'^{B\to C}$ is another CPTNI-preserving supermap. We will prove that $\Theta^{B\to C}$ must be *completely* CPTNI-preserving. In the Choi picture we have

$$\mathbf{J}_{\Theta}^{BC_0} + \mathbf{J}_{\Theta'}^{BC_0} = \mathbf{J}_{\Phi}^{BC_0}. \tag{25}$$

Let us multiply both sides of Eq. (25) by the transpose of any matrix $M^{BC_0} \geq 0$ with marginal $M^{B_0C_0} = I^{B_0}\otimes\rho^{C_0}$, $\rho^{C_0} \in \mathscr{D}\left(\mathscr{H}^{C_0}\right)$, and then take the trace.

$$\operatorname{Tr}\left[\mathbf{J}_{\Theta}^{BC_0}\left(M^{BC_0}\right)^{\mathrm{T}}\right] + \operatorname{Tr}\left[\mathbf{J}_{\Theta'}^{BC_0}\left(M^{BC_0}\right)^{\mathrm{T}}\right] = \operatorname{Tr}\left[\mathbf{J}_{\Phi}^{BC_0}\left(M^{BC_0}\right)^{\mathrm{T}}\right] \tag{26}$$

By the results in Appendix B.2, the right-hand side is 1 because $\Phi^{B\to C}$ is a superchannel. Thus Eq. (26) becomes

$$\operatorname{Tr}\left[\mathbf{J}_{\Theta}^{BC_0}\left(M^{BC_0}\right)^{\mathrm{T}}\right] + \operatorname{Tr}\left[\mathbf{J}_{\Theta'}^{BC_0}\left(M^{BC_0}\right)^{\mathrm{T}}\right] = 1,$$

which implies $\operatorname{Tr}\left[\mathbf{J}_{\Theta}^{BC_0}\left(M^{BC_0}\right)^{\mathrm{T}}\right] \leq 1$ for all M^{BC_0} because $\Theta^{B\to C}$ is CPP. Therefore $\Theta^{B\to C}$ satisfies Eq. (7), which means that it is completely CPTNI-preserving. This concludes the proof. □

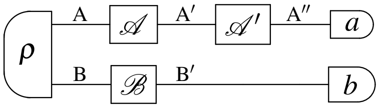
Applying the statement of this theorem to $\Theta'^{B\to C}$, we get that $\Theta'^{B\to C}$ is completely CPTNI-preserving too.

E OPT interpretation of the result

The theory of quantum supermaps, where generic evolutions of quantum maps are described by supermaps, can be analyzed using the framework of operational-probabilistic theories (OPTs) [15, 16, 11, 17, 35, 36, 37], which is a formalism to describe arbitrary physical theories admitting probabilistic processes. OPTs differ from the convex set approach to general probabilistic theories [4, 2, 3] in that they take the composition of physical processes and systems as a primitive. Mathematically, this is based on the graphical language of circuits [24, 25, 52, 26] and probability theory.

E.1 The general framework

OPTs describe the experiments that can be performed on a given set of systems by a given set of physical processes. The framework is based on a primitive notion of composition, whereby every pair of physical systems A and B can be combined into a composite system AB. Physical processes can be connected in sequence or in parallel to build circuits, in the very same way as the corresponding devices are connected in a laboratory to build an experiment. For instance,



. (27)

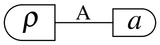
In this example, A, A′, A″, B, and B′ are *systems*, ρ is a bipartite *state*, $\mathscr{A}$, $\mathscr{A}'$ and $\mathscr{B}$ are *transformations*, a and b are *effects*. Note that inputs are on the left and outputs on the right.

For generic systems A and B, we denote by

- $\mathsf{St}(\mathrm{A})$ the set of states of system A,
- $\mathsf{Eff}(\mathrm{A})$ the set of effects on A,
- $\mathsf{Transf}(\mathrm{A},\mathrm{B})$ the set of transformations from A to B,
- $\mathscr{B}\circ\mathscr{A}$ (or $\mathscr{B}\mathscr{A}$, for short) the sequential composition of two transformations $\mathscr{A}$ and $\mathscr{B}$, with the input of $\mathscr{B}$ matching the output of $\mathscr{A}$,
- $\mathscr{I}_{\mathrm{A}}$ the identity transformation on system A, represented by the plain wire $\overset{\mathrm{A}}{\text{——}}$,
- $\mathscr{A}\otimes\mathscr{B}$ the parallel composition (or tensor product) of the transformations $\mathscr{A}$ and $\mathscr{B}$.

Among the list of valid physical systems, every OPT includes the trivial system I, corresponding to the degrees of freedom ignored by theory and to the lack of input (or output) system. States (resp. effects) are transformations with the trivial system as input (resp. output).

A circuit with no external wires, like in Eq. (27), is identified with a real number in the interval $[0,1]$, interpreted as the probability of the joint occurrence of all the transformations present in the circuit. We will often use the notation $(a|\rho)$ to denote the circuit

$(a|\rho)$:=


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Let us clarify these concepts in quantum theory.

Example 5. In quantum theory we associate a Hilbert space $\mathscr{H}^A$ with every system A. States are positive semi-definite operators ρ with $\mathrm{Tr}[\rho]\leq 1$. The reason why we also consider states with trace less than 1 will be explained in example 7. An effect is, instead, represented by a positive semi-definite operator E, with $E\leq I$, where I is the identity operator. The pairing between states and effects is given by the trace: $(E|\rho)=\mathrm{Tr}\,E\rho$.

The fact that some circuits represent real numbers induces a notion of sum for transformations, so that the sets $\mathsf{St}(\mathrm{A})$, $\mathsf{Transf}(\mathrm{A},\mathrm{B})$, and $\mathsf{Eff}(\mathrm{A})$ become spanning sets of *real* vector spaces. We will denote the vector space of states as $\mathsf{St}_{\mathbb{R}}(\mathrm{A})$ and the vector space of transformations as $\mathsf{Transf}_{\mathbb{R}}(\mathrm{A},\mathrm{B})$. Effects become linear functionals on $\mathsf{St}_{\mathbb{R}}(\mathrm{A})$, and transformations in $\mathsf{Transf}(\mathrm{A},\mathrm{B})$ are linear transformations from $\mathsf{St}_{\mathbb{R}}(\mathrm{A})$ to $\mathsf{St}_{\mathbb{R}}(\mathrm{B})$.

If we restrict ourselves to linear combinations of states with non-negative coefficients (conical combinations), we obtain a proper convex cone [15], called the cone of states $\mathsf{St}_{+}(\mathrm{A})$. Note that effects take non-negative values on the cone of states. Indeed if $\xi \in \mathsf{St}_+(\mathrm{A})$, then ξ is a conical combination of some states ρ_i: $\xi = \sum_i \lambda_i \rho_i$, where $\lambda_i \geq 0$ for every i. Therefore when an effect a acts on ξ, we have $(a|\xi) = \sum_i \lambda_i (a|\rho_i) \geq 0$ as $\lambda_i \geq 0$, and $0 \leq (a|\rho_i) \leq 1$, because an effect yields a probability when applied to a state.

Example 6. In quantum theory, $\mathsf{St}_{\mathbb{R}}(\mathrm{A})$ is the vector space of hermitian operators on $\mathscr{H}^A$, and $\mathsf{St}_+(\mathrm{A})$ is the cone of positive semi-definite operators.

In general, an experiment in a laboratory can be non-deterministic, i.e. it can result into a set of alternative transformations applied to the input system, heralded by different outcomes, which can (at least in principle) be accessed by an experimenter. General non-deterministic processes are described by *tests*: a test from A to B is a collection of transformations $\{\mathscr{C}_x\}_{x\in X}$ from A to B, where X is the set of outcomes. If A (resp. B) is the trivial system, the test is called a *preparation-test* (resp. *observation-test*). If the set of outcomes X contains a single element, we say that the test is *deterministic*, because only one transformation can occur, and we can predict the outcome of the experiment. We refer to deterministic transformations as *channels*. If we sum over all the transformations in a test we get a deterministic transformation, viz. a channel: $\mathscr{C} := \sum_{x\in X} \mathscr{C}_x$. This is because the sum of all the transformations arising in a test can be viewed as the full coarse-graining over all outcomes [15], resulting in a new, deterministic, test.

Example 7. In quantum theory, a channel from A_0 to A_1 is a CPTP map from $\mathscr{B}\left(\mathscr{H}^{A_0}\right)$ to $\mathscr{B}\left(\mathscr{H}^{A_1}\right)$. A test from A_0 to A_1 is a collection of CPTNI maps from $\mathscr{B}\left(\mathscr{H}^{A_0}\right)$ to $\mathscr{B}\left(\mathscr{H}^{A_1}\right)$ summing to a CPTP map. Note that this is consistent with the fact that the sum over all the transformations in a test yields a channel.

Deterministic states are positive semi-definite operators ρ with $\mathrm{Tr}[\rho] = 1$. A non-deterministic preparation-test is a collection of positive semi-definite operators ρ_i with $\mathrm{Tr}[\rho_i] < 1$ (non-deterministic states) that sum to a deterministic state ρ. This is essentially a random preparation: a state ρ_i is prepared with a probability given by $\mathrm{Tr}[\rho_i]$. This is why we consider all positive semi-definite operators ρ with $\mathrm{Tr}[\rho] \leq 1$ as states.

Observation-tests are POVMs. In quantum theory there is only one deterministic effect: the identity I (more precisely it is the functional $\mathrm{Tr}[I\bullet]$). This is not a coincidence, but it follows from the fact that quantum theory is a causal theory (see definition 8).

Among all theories, *causal* theories [15] are particularly important: in these theories, loosely speaking, information cannot come back from the future. They are particularly simple in their structure, and generally speaking they are well understood. Causality can also be shown to imply no-signaling in space-like separated systems [15]. The formal statement of the property of causality is as follows.

Axiom 8 (Causality [15])**.** *For every state ρ, take two observation-tests $\{a_x\}_{x\in X}$ and $\{b_y\}_{y\in Y}$. One has*

$$\sum_{x\in X} (a_x|\rho) = \sum_{y\in Y} (b_y|\rho) .$$

Causality can be equivalently characterized in terms of deterministic effects: an OPT is *causal* if and only if, for every system A, there is a unique deterministic effect u_{A} [15]. This characterization is very practical to work with.

Example 9. In quantum theory there is only one deterministic effect, the identity operator (or the trace functional). Hence quantum theory is causal.

Causal theories enjoy an important property: the unique deterministic effect for a composite system AB always factorizes as the parallel composition of the deterministic effects on A and on B. In symbols, $u_{\mathrm{AB}} = u_{\mathrm{A}} \otimes u_{\mathrm{B}}$. This is because if u_{A} and u_{B} are the deterministic effects of A and B, then $u_{\mathrm{A}} \otimes u_{\mathrm{B}}$ is a deterministic effect on AB. Since the theory is causal, there is a unique deterministic effect on AB, so $u_{\mathrm{A}} \otimes u_{\mathrm{B}}$ is *the* deterministic effect of AB.

Moreover in causal theories there is a nice characterization of channels: a transformation $\mathscr{C} \in \mathsf{Transf}(\mathrm{A}, \mathrm{B})$ is a channel if and only if [15]

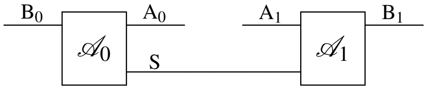
$$u_{\mathrm{B}} \mathscr{C} = u_{\mathrm{A}}. \tag{28}$$

In quantum theory, since u is the trace, this condition amounts to saying that channels are trace-preserving.

Let us conclude this section showing how the theory of quantum supermaps fits into the OPT formalism.

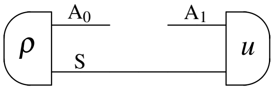
Example 10. In the theory of quantum supermaps, every system A is a pair of input and output quantum systems $\mathrm{A} = (A_0, A_1)$; deterministic states are CPTP maps, and non-deterministic ones are CPTNI maps. The cone of states is given by all CP maps. Transformations in this theory are supermaps [13, 18, 47, 10, 31]. As our results show, it is not immediate to pin down the mathematical properties that make a generic linear supermap from A to B physical. We will analyze this issue from the OPT perspective in the next subsection.

Now let us show that the theory of quantum supermaps is *not* causal. Suppose we want to construct a deterministic effect in this theory. According to [12, 13, 14], to this end it is enough to consider a 1-comb made of deterministic quantum operations, which means a circuit fragment of the form



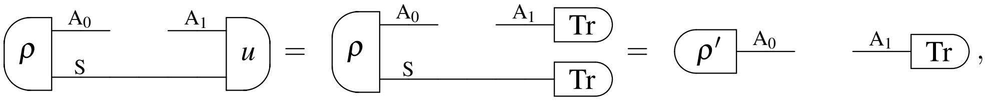
,

where both $\mathscr{A}_0$ and $\mathscr{A}_1$ are deterministic quantum operations. Since this comb must output a probability, its pre-processing $\mathscr{A}_0$ must be a deterministic bipartite quantum state $\rho \in \mathsf{St}(\mathrm{A}_0\mathrm{S})$, and its post-processing $\mathscr{A}_1$ must be a deterministic bipartite quantum effect $u \in \mathsf{Eff}(\mathrm{A}_1\mathrm{S})$, for some system S:



.

Now recall that in causal theories the deterministic effect of a bipartite system $\mathrm{A}_1\mathrm{S}$ factorizes as $u_{\mathrm{A}_1} \otimes u_{\mathrm{S}}$, and that u is nothing but the trace (cf. example 7). Then



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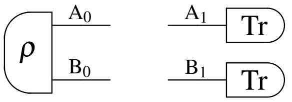
where $\rho' = \mathrm{Tr}_{\mathrm{S}}[\rho]$. In this way, for any choice of $\rho \in \mathscr{D}\left(\mathscr{H}^{A_0 S}\right)$ we obtain *all* quantum states $\rho' \in \mathscr{D}\left(\mathscr{H}^{A_0}\right)$. Therefore the generic deterministic effect on system $\mathrm{A} = (A_0, A_1)$ of the theory of quantum supermaps is of the form

$u_\rho =$ 

,

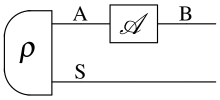
for any quantum state $\rho \in \mathscr{D}\left(\mathscr{H}^{A_0}\right)$. This means that there is a whole family of deterministic effects, labeled by quantum states. Therefore the theory of quantum supermaps is *not* causal, a fact that is confirmed by the presence of signaling bipartite quantum channels [7]. The failure of causality implies here that there are some deterministic effects for a bipartite system $\mathrm{AB} = (A_0, A_1)(B_0, B_1)$ that do not factorize. Indeed, if we take a non-product bipartite quantum state $\rho \in \mathscr{D}\left(\mathscr{H}^{A_0B_0}\right)$, the associated deterministic effect is

$u_\rho =$  , (29)

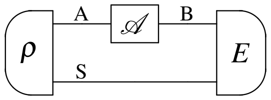
which does not factorize. This fact will play an important role in Appendix E.2, and it is ultimately the reason why we need the CPTNI preservation condition in a complete sense.

E.2 Necessary conditions for physical transformations

In the OPT approach, however we construct a diagram, this represents a physical object: a valid state, a valid transformation, a valid effect. Specializing our analysis to transformations from a system A to a system B, a linear map $\mathscr{A}$ from $\mathsf{St}_{\mathbb{R}}(\mathrm{A})$ to $\mathsf{St}_{\mathbb{R}}(\mathrm{B})$ is a valid physical transformation if and only if

 (30)

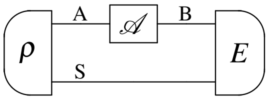
is a valid state of system BS, for every choice of ρ and S. Here we will derive some *necessary* conditions to guarantee this. In particular if (30) is a valid state, for every bipartite effect $E \in \mathsf{Eff}(\mathrm{BS})$ we have

$0 \leq$  $\leq 1,$ (31)

because this is the probability of E occurring on $(\mathscr{A} \otimes \mathscr{I}_{\mathrm{S}})\,\rho$.

Remark 11. Condition (31) is only *necessary*, but in general *not sufficient* to guarantee that (30) represents a valid physical state. Indeed, the theory may have additional restrictions on the allowed states, as it happens in the presence of superselection rules [41, 28, 21, 22, 51]. If the theory is completely unrestricted, like quantum theory or the theory of quantum supermaps, condition (31) is sufficient as well.

Let us analyze the two inequalities in (31) separately. If $(\mathscr{A} \otimes \mathscr{I}_{\mathrm{S}})\,\rho$ is in the cone of states of BS, then we immediately have

 $\geq 0,$

for every effect $E \in \mathsf{Eff}(\mathrm{BS})$.

Definition 12. We say that a transformation $\mathscr{A}$ in $\mathsf{Transf}_{\mathbb{R}}(\mathrm{A},\mathrm{B})$ is *completely positive* if, for every system S and every element $\xi \in \mathsf{St}_{+}(\mathrm{AS})$, we have $(\mathscr{A} \otimes \mathscr{I}_{\mathrm{S}})\,\xi \in \mathsf{St}_{+}(\mathrm{BS})$.

In words, a completely positive transformation is a linear transformation that maps elements in the input cone of states to elements in the output cone of states in a complete sense, i.e. even when there is an ancillary system S. This is clearly a necessary condition for a transformation to be physical.

Note that it is equivalent to define complete positivity just on states in $\mathsf{St}(\mathrm{AS})$, instead of on generic elements of $\xi \in \mathsf{St}_{+}(\mathrm{AS})$: $\mathscr{A}$ is completely positive if and only if, for every system S and every state $\rho \in \mathsf{St}(\mathrm{AS})$, we have $(\mathscr{A} \otimes \mathscr{I}_{\mathrm{S}})\,\rho \in \mathsf{St}_{+}(\mathrm{BS})$. To see the non-trivial implication, recall that if ξ is a generic element of $\mathsf{St}_+(\mathrm{AS})$, it can be written as a conical combination of states ρ_i of AS: $\xi = \sum_i \lambda_i \rho_i$, with $\lambda_i \geq 0$ for every i. Then, if we know that $(\mathscr{A} \otimes \mathscr{I}_{\mathrm{S}})\rho \in \mathsf{St}_+(\mathrm{BS})$ for every $\rho \in \mathsf{St}(\mathrm{AS})$, we have

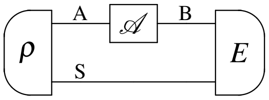
$$(\mathscr{A} \otimes \mathscr{I}_{\mathrm{S}})\xi = \sum_i \lambda_i (\mathscr{A} \otimes \mathscr{I}_{\mathrm{S}})\rho_i \in \mathsf{St}_+(\mathrm{BS}),$$

because $\mathsf{St}_+(\mathrm{BS})$ is closed under conical combinations.

Example 13. In quantum theory, the cone of states is the cone of positive semi-definite operators; therefore completely positive transformations in the sense of definition 12 are exactly CP maps.

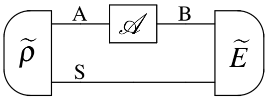
In the theory of quantum supermaps, the cone of states is the cone of CP maps. In this case, completely positive transformations are CPP supermaps [13, 31].

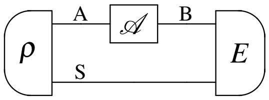
Now let us analyze the second inequality in (31), namely

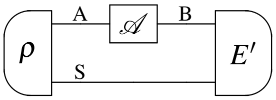


$\leq 1,$ (32)

for every effect $E \in \mathsf{Eff}(\mathrm{BS})$. Assume $\mathscr{A}$ is completely positive. Then, demanding the validity of inequality (32) for every state $\rho \in \mathsf{St}(\mathrm{AS})$ and every effect $E \in \mathsf{Eff}(\mathrm{BS})$ is equivalent to demanding its validity when ρ is any *deterministic* state and E any *deterministic* effect. To see the non-trivial implication, recall that if ρ is non-deterministic, it arises in a preparation-test $\{\rho, \rho'\}$. Similarly, if E is non-deterministic, it arises in an observation-test $\{E, E'\}$. Clearly $\widetilde{\rho} = \rho + \rho'$ is a deterministic state, and $\widetilde{E} = E + E'$ is a deterministic effect. Then

$1 \geq$ 

$=$ 

$+$ 

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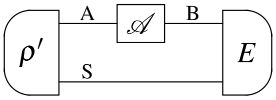
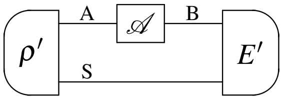
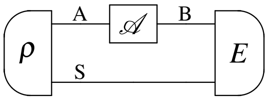
$+$ 

$+$ 

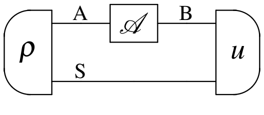
Now, each term in the right-hand side is non-negative because $\mathscr{A}$ is completely positive. It follows that each term is also less than or equal to 1, and specifically



$\leq 1.$

We summarize these necessary requirements in the following theorem.

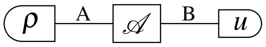
Theorem 14. *Let* $\mathscr{A} \in \mathsf{Transf}_{\mathbb{R}}(\mathrm{A}, \mathrm{B})$. *Then* $\mathscr{A}$ *is a physical transformation only if both these conditions are satisfied:*

1. $(\mathscr{A} \otimes \mathscr{I}_{\mathrm{S}})\rho \in \mathsf{St}_+(\mathrm{BS})$ *for every system* S *and every state* $\rho \in \mathsf{St}(\mathrm{AS})$;
2. 

$\leq 1,$

   for every system S, *every* deterministic *state* $\rho \in \mathsf{St}(\mathrm{AS})$, *and every* deterministic *effect* $u \in \mathsf{Eff}(\mathrm{BS})$.

Note that in particular, condition 2 implies that



$$\leq 1, \tag{33}$$

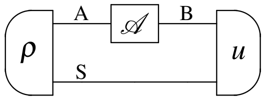
for it is enough to take S to be the trivial system I. However, in general, this condition is *weaker* than condition 2, such as in the theory of quantum supermaps. Let us analyze the role of conditions 1, 2, and (33) in this theory.

Example 15. First of all, since the theory of quantum supermaps has no restrictions, the conditions in theorem 14 become *sufficient* as well. We have already examined condition 1. Let us focus on condition 2, and unfold its meaning. In this case, ρ is actually bipartite channel $\mathscr{C}$, and $\mathscr{A}$ acts as a supermap $\mathfrak{A}$ on half of $\mathscr{C}$. Recalling Eq. (29), condition 2 becomes $\mathrm{Tr}_{BS}\left(\mathfrak{A}^{A\to B}\otimes\mathfrak{I}^{S}\right)\left[\mathscr{C}^{AS}\left(\rho^{A_0S_0}\right)\right]\leq 1$, where $\mathfrak{I}$ is the identity superchannel. This is nothing but requiring that $\mathfrak{A}$ be completely CPTNI-preserving (cf. Eq. (5)).

In conclusion, the two conditions of theorem 14 are exactly the two conditions we found in this letter. Note that condition (33), expressing CPTNI preservation (but not in a complete sense), is *weaker* than condition 2, as there is no way to recover condition 2 from condition (33). This is essentially because not all bipartite deterministic effects can be reduced to single-system deterministic effects (cf. Eq. (29)). Thus condition (33) *cannot* be used to assess whether a candidate supermap is physical or not, and CPTNI preservation is not enough.

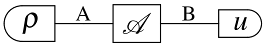
If theorem 14 is valid in all physical theories, why do we not need to impose the trace non-increasing condition in a complete sense in quantum theory? This is because the theory is causal. Indeed in all causal theories, condition 2 becomes equivalent to condition (33).

Proposition 16. *In a causal theory with deterministic effect u, one has*



$$\leq 1,$$

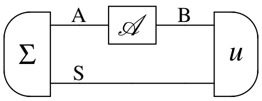
for every system S *and every deterministic state* $\rho\in\mathsf{St}(\mathrm{AS})$*, if and only if*



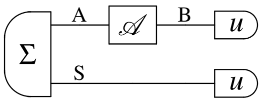
$$\leq 1.$$

for every deterministic state $\rho\in\mathsf{St}(\mathrm{A})$.

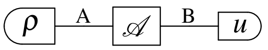
Proof. We have already seen one implication (necessity), now let us focus on the other. Assume condition (33) holds. Take an arbitrary system S and an arbitrary deterministic state $\Sigma\in\mathsf{St}(\mathrm{AS})$. Then



$$=$$

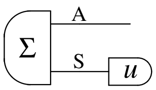


$$=$$

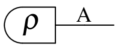


$$\leq 1,$$

where we have used the fact that the deterministic effect of a composite system factorizes, and that



$$=:$$



is a deterministic state. $\square$

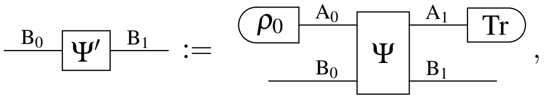
In other words, for causal theories condition 2 can be formulated only for single systems, without the need of an ancillary system S. Recall that in quantum theory u is the trace, so condition (33) means that $\mathscr{A}$ is trace-non-increasing. Proposition 16 is the ultimate reason why in quantum theory it is enough to require that a CP map be TNI (on single system) rather than *completely* TNI. In conclusion, the ultimate origin of the unexpected behavior of the theory of quantum supermaps is the failure of causality.

However, in [31] one of the authors showed that for a CPP map to be a superchannel, instead, it is not necessary to demand that it be completely TPP, but it is enough that it be TPP. Why do we not need CPTP preservation in a complete sense for superchannels? Let us understand it using the OPT formalism.

Clearly, a superchannel $\Theta^{B\to C}$ must send channels to channels in a complete sense: for any bipartite quantum channel Ψ^{AB}, $\mathscr{I}^A \otimes \Theta^{B\to C}\left[\Psi^{AB}\right] = \Phi^{AC}$, where $\mathscr{I}^A$ is the identity superchannel, and Φ^{AC} is still a quantum channel. By Eq. (28), this is true if and only if

$$\left(\mathrm{Tr}_{A_1} \otimes \mathrm{Tr}_{C_1}\right)\left(\mathscr{I}^A \otimes \Theta^{B\to C}\left[\Psi^{AB}\right]\right) = \mathrm{Tr}_{A_0} \otimes \mathrm{Tr}_{C_0}, \tag{34}$$

where we have denoted the deterministic effect u explicitly as the trace. Now let us try to prove Eq. (34) knowing that $\Theta^{B\to C}$ is *just* TPP. Now consider the following channel



(35)

where ρ_0 is some density matrix on A_0. Since $\Theta^{B\to C}$ is TPP, we have that $\Phi'^{C} := \Theta^{B\to C}\left[\Psi'^{B}\right]$ is still a quantum channel. In other words

$$\mathrm{Tr}_{C_1}\Theta^{B\to C}\left[\Psi'^{B}\right] = \mathrm{Tr}_{C_0}.$$

Then if we take a density matrix $\sigma_0 \in \mathscr{D}\left(\mathscr{H}^{C_0}\right)$, we have

$$\mathrm{Tr}_{C_1}\Theta^{B\to C}\left[\Psi'^{B}\right]\left(\sigma_0^{C_0}\right) = \mathrm{Tr}_{C_0}\left[\sigma_0^{C_0}\right] = 1.$$

Now, recalling the definition of Ψ'^{B} in Eq. (35), we have

$$\mathrm{Tr}_{A_1}\mathrm{Tr}_{C_1}\left(\mathscr{I}^A \otimes \Theta^{B\to C}\left[\Psi^{AB}\right]\right)\left(\rho_0^{A_0} \otimes \sigma_0^{C_0}\right) = 1, \tag{36}$$

for any $\rho_0 \in \mathscr{D}\left(\mathscr{H}^{A_0}\right)$ and any $\sigma_0 \in \mathscr{D}\left(\mathscr{H}^{C_0}\right)$. If we manage to prove that

$$\mathrm{Tr}_{A_1}\mathrm{Tr}_{C_1}\left(\mathscr{I}^A \otimes \Theta^{B\to C}\left[\Psi^{AB}\right]\right)\left(\tau^{A_0C_0}\right) = 1$$

for every bipartite state $\tau^{A_0C_0}$, then the validity of Eq. (34) is shown. Now, recall that in quantum theory every bipartite state can be written as an affine combination of product states. Therefore $\tau^{A_0C_0} = \sum_j \lambda_j \rho_j^{A_0} \otimes \sigma_j^{C_0}$, with $\sum_j \lambda_j = 1$. Therefore

$$\mathrm{Tr}_{A_1}\mathrm{Tr}_{C_1}\left(\mathscr{I}^A \otimes \Theta^{B\to C}\left[\Psi^{AB}\right]\right)\left(\tau^{A_0C_0}\right) = \mathrm{Tr}_{A_1}\mathrm{Tr}_{C_1}\left(\mathscr{I}^A \otimes \Theta^{B\to C}\left[\Psi^{AB}\right]\right)\left(\sum_j \lambda_j \rho_j^{A_0} \otimes \sigma_j^{C_0}\right) =$$

$$= \sum_j \lambda_j \mathrm{Tr}_{A_1}\mathrm{Tr}_{C_1}\left(\mathscr{I}^A \otimes \Theta^{B\to C}\left[\Psi^{AB}\right]\right)\left(\rho_j^{A_0} \otimes \sigma_j^{C_0}\right) = \sum_j \lambda_j = 1,$$

where we have used Eq. (36). This proves Eq. (34), so for quantum superchannels it is indeed enough to require that they be TPP. Note that this proof does not use any quantum feature except causality, which allows us to characterize quantum channels as CPTP maps, and local tomography [15, 37], a property that guarantees that every deterministic bipartite state can be written as an affine combination of deterministic product states.

The same proofs also shows that any attempt to adapt it to supermaps transforming quantum channels to CPTNI maps is bound to fail: even if $\mathrm{Tr}_{A_1}\mathrm{Tr}_{C_1}\left(\mathscr{I}^A\otimes\Theta^{B\to C}\left[\Psi^{AB}\right]\right)\left(\rho_0^{A_0}\otimes\sigma_0^{C_0}\right)\leq 1$, we cannot conclude that $\mathrm{Tr}_{A_1}\mathrm{Tr}_{C_1}\left(\mathscr{I}^A\otimes\Theta^{B\to C}\left[\Psi^{AB}\right]\right)\left(\tau^{A_0C_0}\right)\leq 1$ for every bipartite state $\tau^{A_0C_0}$. The reason is that we are only dealing with an *affine* combination, possibly even containing negative terms. This does not allow us to conclude anything about $\sum_j\lambda_j\mathrm{Tr}_{A_1}\mathrm{Tr}_{C_1}\left(\mathscr{I}^A\otimes\Theta^{B\to C}\left[\Psi^{AB}\right]\right)\left(\rho_j^{A_0}\otimes\sigma_j^{C_0}\right)$.