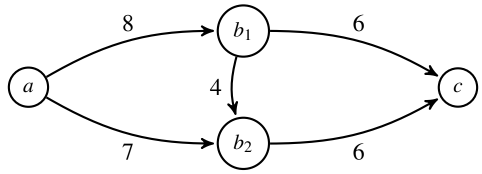
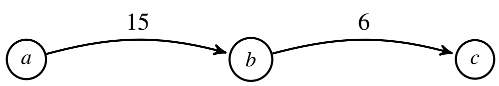
Networks and dynamical systems are finding their way into nearly every branch of science and their usefulness for describing physical processes is well established. One particular kind of dynamical system is given by a Markov process - here, we will only consider Markov processes with a finite set of states and whose transition rates are constant. These kinds of Markov processes are sometimes known as *finite Markov chains* and an example is given by the following:

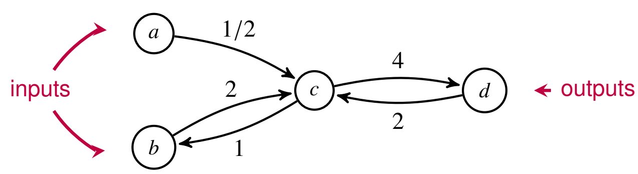


This is a very simple example with only four states. In practice, these Markov processes can have hundreds or even thousands of states and be computationally cumbersome to work with. One method of 'simplifying' a Markov process with many states into a smaller one with fewer states whose dynamics approximates that of the larger one is 'coarse-graining'. Coarse-grainings can come in many different flavors, but one of the more well known notions of coarse-grainings for Markov processes if given by 'lumpability' [4]. Lumpability roughly means grouping the states of the larger open Markov process in such a way that the groups, or *lumps*, behave as the individual states that constitute them. For example, the above Markov process is indeed lumpable with its coarse-graining given by the following simpler Markov process:



Here, the idea is that it does not matter which state b_1 or b_2 we are in as they both transition to state c with a rate of 6, nor does it matter which route from state a we took to get to state b_1 or b_2, and so we can combine states b_1 and b_2 into a single state b.

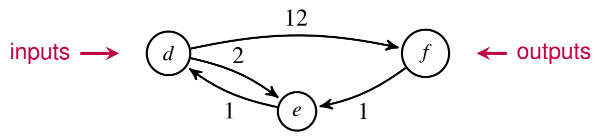
There is more that we can do with Markov processes of the above sort. We can consider them as 'open' systems by prescribing certain states to be inputs and certain states to be outputs. For example:



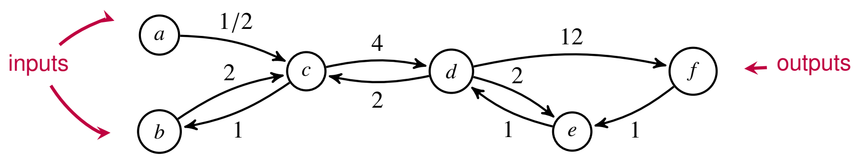
Here we have a Markov process with a finite set of states given by $\{a,b,c,d\}$ and we interpret the subset $\{a,b\}$ as inputs and the singleton $\{d\}$ as the output, making the above Markov process into an *open* Markov process. This can be made rigorous with the idea of a *cospan*, which is a diagram in any category of the form:

$$S \xrightarrow{i} X \xleftarrow{o} T$$

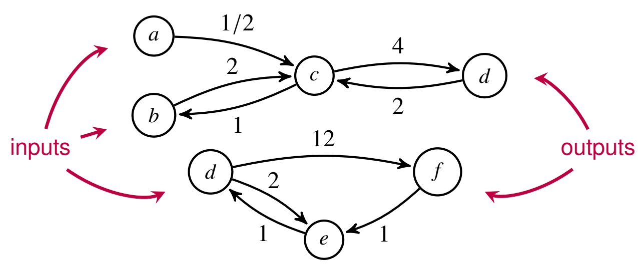
This viewpoint naturally leads to thinking of open Markov processes as morphisms in some kind of category, and if we denote the above open Markov process by M, we can think of this process as a morphism $M\colon \{a,b\} \to \{d\}$. We can then compose the above open Markov process with the following one:



Denoting this open Markov process by $N\colon \{d\} \to \{f\}$, we can compose these in a visually obvious manner by identifying, or *pushing out over*, the common state d of both open Markov processes and the result is the following open Markov process $M \odot N\colon \{a,b\} \to \{f\}$.



We can also consider these open Markov processes in parallel which suggests an underlying monoidal structure:



This would be an open Markov process $M+N\colon \{a,b\}+\{d\} \to \{d\}+\{f\}$. All of this ties up nicely into one neat package - in a recent work with John Baez [3], we prove there exists a *double category* of open Markov processes. Double categories were first introduced by Ehresmann [5, 6], and they have long been used in topology and other branches of pure mathematics. As the name suggests, double categories have 2-dimensional morphisms, or 2-morphisms, which look like this:

$$\begin{array}{ccc} A & \xrightarrow{M} & B \\ {\scriptstyle f}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle g} \\ C & \xrightarrow[N]{} & D \end{array}$$

While a mere category has only objects and morphisms, here we have a few more types of entities. We call A,B,C and D 'objects', f and g 'vertical 1-morphisms', M and N 'horizontal 1-cells', and α a '2-morphism'. We can compose vertical 1-morphisms to get new vertical 1-morphisms and compose horizontal 1-cells to get new horizontal 1-cells. We can compose the 2-morphisms in two ways: horizontally by setting squares side by side, and vertically by setting one on top of the other. In a 'strict' double category all these forms of composition are associative. In a 'pseudo' double category, horizontal 1-cells compose in a weakly associative manner: that is, the associative law holds only up to an invertible 2-morphism, called the 'associator', which obeys a coherence law. This is just a quick sketch of the ideas; for full definitions see for example the works of Grandis and Paré [7, 8]. We prove:

Theorem 1. *There exists a (pseudo) double category* $\mathbb{M}\mathbf{ark}$ *which has:*

(i) finite sets as objects,
(ii) maps between finite sets as vertical 1-morphisms,
(iii) open Markov processes as horizontal 1-cells,
(iv) morphisms between open Markov processes given by coarse-grainings as 2-morphisms.

For the details regarding the structure of this double category, such as how horizontal 1-cells or 2-morphisms are composed, we refer to Section 4 of the paper [3].

Composition of open Markov processes is only weakly associative as it involves taking pushouts, so this is a pseudo double category. Moreover, this double category is symmetric monoidal in the sense of Shulman [9].

One can choose from many different avenues as to what they want these open Markov processes to serve as a model for, or what about them they wish to study. In a previous work, Baez, Fong and Pollard studied the relation between probabilities and flows at the inputs and outputs that holds while in a steady state [1, 2]. They called the process of extracting this relation from an open Markov process 'black-boxing', since it gives a way to forget the internal workings of an open system and remember only its externally observable behavior. They proved that black-boxing is compatible with the above composition and tensoring, which can be summarized by saying that black-boxing is a symmetric monoidal functor

$$\blacksquare\colon \mathsf{Mark} \to \mathsf{LinRel}.$$

The main result of our work [3] is that this black-boxing functor can be extended to a *double functor*

$$\blacksquare\colon \mathbb{M}\mathbf{ark} \to \mathbb{L}\mathbf{inRel}$$

showing that black-boxing as defined by Baez, Fong and Pollard is compatible with our choice of coarse-grainings.

Theorem 2. *There exists a symmetric monoidal double functor* $\blacksquare\colon \mathbb{M}\mathbf{ark} \to \mathbb{L}\mathbf{inRel}$.

For the details on how this double functor is defined, we refer to Theorem 5.5 of the paper [3].

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