Categorical Equivalence between Orthocomplemented Quantales and Complete Orthomodular Lattices (extended abstract of [4])

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1 Introduction

Different quantum structures emphasize different aspects of quantum reasoning. In this paper, we focus on two types of quantum structures. One structure is a complete orthomodular lattice, whose points are viewed as testable properties of a quantum system. While these structures are natural settings for reasoning about the properties of a system, they leave implicit important quantum structures, such as quantum actions. The other structure is the quantum dynamic algebra, a quantale augmented with an orthocomplementation. A quantale is a complete lattice with an additional semi-group (often a monoidal) operator, and this additional operator is viewed as function composition, with the lattice points abstractions of functions. In this way, quantales are often described as abstractions of operator algebras such as C^*-algebras, though their connection to orthomodular lattices or even the more restrictive Hilbert lattices (lattice of closed subspaces of a Hilbert space) has received little attention. Quantum dynamic algebras were proposed in [1] as providing a type of algebraic quantum structure that makes actions first-class citizens. While the primary purpose of these structures in that work was to connect them to quantum dynamic frames, a graph-like counterpart to the Hilbert lattice, their paper did more to suggest connections than to establish rigorous equivalence, and it did not involve categorical structure.

This paper establishes a categorical equivalence between complete orthomodular lattices and orthomodular dynamic algebras, a type of quantum dynamic algebra. This equivalence clarifies how dynamics arises from Hilbert lattices, and establishes a clearer connection between quantales and orthomodular lattices, thus clarifying how they can be used for quantum reasoning. For this purpose we construct a pair of functors from each category to the other such that the composition of the two functors (composed in either way) is naturally isomorphic to the corresponding identity functor. Categorical equivalences and dualities such as this have been extensively studied in the literature, for example those between quantum geometries and quantum lattices (between Hilbert geometries and propositional systems, as well as projective geometries and projective lattices), as given in [6], or categorical dualities include those between quantum lattices and quantum graph-like structures (between Piron lattices and quantum dynamic algebras), as given in [2]. These results allow us to establish meaningful relationships among different structures and to help transfer results about one structure to results about another.

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2 The Categories

2.1 The Category $\mathbb{L}$ of Complete Orthomodular Lattices

In an ortho-lattice $\mathfrak{L} = (L, \leq, -^{\perp})$, for each $p \in L$, we can define an important pair of operations called the *Sasaki projection (onto p)* and *Sasaki hook (from p)* [5]:

$$f_p : L \to L :: q \mapsto p \wedge (p^{\perp} \vee q), \qquad f^p : L \to L :: q \mapsto p^{\perp} \vee (p \wedge q).$$

A crucial fact about this pair of order-preserving maps is that $\mathfrak{L}$ is orthomodular if and only if, for every $p \in L$, f_p is left adjoint to f^p [3]. An *ortho-lattice isomorphism*, or $\mathbb{L}$*-morphism*, from an ortho-lattice $\mathfrak{L}_1 = (L_1, \leq_1, -^{\perp_1})$ to $\mathfrak{L}_2 = (L_2, \leq_2, -^{\perp_2})$ is a function $k : L_1 \to L_2$ such that, for any $p_1, q_1 \in L_1$,

1. k is a bijection; 2. $p_1 \leq_1 q_1 \Leftrightarrow k(p_1) \leq_2 k(q_1)$; 3. $k(p_1^{\perp_1}) = (k(p_1))^{\perp_2}$.

Complete orthomodular lattices equipped with ortho-lattice isomorphisms form a category $\mathbb{L}$.

2.2 The Category $\mathbb{Q}$ of Orthomodular Dynamic Algebras

A *generalized dynamic algebra* is a tuple $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$ consisting of a non-empty set Q equipped with functions $\bigsqcup : \wp(Q) \to Q$, $\cdot : Q \times Q \to Q$ and $\sim : Q \to Q$. We can define the following constructions on a generalized dynamic algebra.

$$\sqsubseteq \stackrel{\text{def}}{=} \left\{(x,y) \in Q \times Q \mid \bigsqcup\{x,y\} = y\right\} \qquad \mathscr{P}_{\mathfrak{Q}} \stackrel{\text{def}}{=} \{\sim x \mid x \in Q\}$$

$$\preceq \stackrel{\text{def}}{=} \left\{(p,q) \in \mathscr{P}_{\mathfrak{Q}} \times \mathscr{P}_{\mathfrak{Q}} \mid \bigvee\{p,q\} = q\right\} \qquad \bigvee X \stackrel{\text{def}}{=} \sim\sim\bigsqcup X, \text{ for } X \subseteq \mathscr{P}_{\mathfrak{Q}}$$

$$\ulcorner x \urcorner \stackrel{\text{def}}{=} \lambda y.\sim\sim(x \cdot y) \qquad \bigwedge X \stackrel{\text{def}}{=} \sim\bigsqcup\{\sim x \mid x \in X\}, \text{ for } X \subseteq \mathscr{P}_{\mathfrak{Q}}$$

$$\equiv \stackrel{\text{def}}{=} \{(x,y) \mid \ulcorner x \urcorner(p) = \ulcorner y \urcorner(p), \text{ for } p \in \mathscr{P}_{\mathfrak{Q}}\}$$

We let $\mathscr{T}_{\mathfrak{Q}} \stackrel{\text{def}}{=} \{x \in Q \mid x = p_1 \cdot \cdots \cdot p_n, \text{ for some } n \in \mathbb{N}^+ \text{ and } p_1, \ldots, p_n \in \mathscr{P}_{\mathfrak{Q}}\}$ be the smallest subset of Q containing $\mathscr{P}_{\mathfrak{Q}}$ which is closed under the operation $\cdot$, and will drop the subscript $\mathfrak{Q}$ when clear. An *orthomodular dynamic algebra* is a generalized dynamic algebra $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$ such that

4. $(Q, \sqsubseteq, \cdot)$ is a quantale, and $\bigsqcup$ is the arbitrary join.
5. $(\mathscr{P}, \preceq, \sim)$ is a complete orthomodular lattice;
6. If X is such that, $\mathscr{P} \subseteq X \subseteq Q$ and X is closed under the operation $\cdot$ and $\bigsqcup$, then $X = Q$ (minimality);
7. for any $X, Y \subseteq \mathscr{T}$, $\bigsqcup X = \bigsqcup Y$, if and only if $X = Y$ (sets);
8. for any $x, y \in \mathscr{T}$, $x = y$ if and only if $x \equiv y$ (completeness);
9. for any $p, q \in \mathscr{P}$, $\ulcorner p \urcorner(q) = f_p(q)$, i.e. $\sim\sim(p \cdot q) = p \wedge (\sim p \vee q)$ (Sasaki projection);
10. $\ulcorner x \urcorner(y) = \ulcorner x \urcorner(\sim\sim y)$, for each $x, y \in Q$ (composition).

A function $\theta : Q_1 \to Q_2$ is a $\mathbb{Q}$*-morphism* from an orthomodular dynamic algebra $\mathfrak{Q}_1 = (Q_1, \bigsqcup_1, \cdot_1, \sim_1)$ to $\mathfrak{Q}_2 = (Q_2, \bigsqcup_2, \cdot_2, \sim_2)$, if:

11. θ restricted to $\mathscr{P}_1 = \{\sim x \mid x \in Q_1\}$ is an ortho-lattice isomorphism from $(\mathscr{P}_1, \preceq_1, \sim_1)$ to $(\mathscr{P}_2, \preceq_2, \sim_2)$;
12. θ preserves $\bigsqcup$ and $\cdot$, i.e. for any $A_1 \subseteq \mathscr{Q}_1$ and $x_1, y_1 \in \mathscr{Q}_1$, $\theta(\bigsqcup_1 A_1) = \bigsqcup_2 \theta[A_1]$ and $\theta(x_1 \cdot_1 y_1) = \theta(x_1) \cdot_2 \theta(x_2)$

Orthomodular dynamic algebras equipped with $\mathbb{Q}$-morphisms form a category $\mathbb{Q}$.

3 The Equivalence

To establish the equivalence, we define functors $\mathbf{F} : \mathbb{L} \to \mathbb{Q}$ and $\mathbf{U} : \mathbb{Q} \to \mathbb{L}$ and natural isomorphisms $\tau : 1_{\mathbb{L}} \to \mathbf{U} \circ \mathbf{F}$ and $\eta : 1_{\mathbb{Q}} \to \mathbf{F} \circ \mathbf{U}$. The functor $\mathbf{F}$ builds, from a complete orthomodular lattice, an orthomodular dynamic algebra of sets of functions on the input lattice, while mapping each morphism k to a function on sets of functions that conjugates each element of the input by k. The functor $\mathbf{U}$ collapses a quantum dynamic algebra $\mathfrak{Q}$ to its set $\mathscr{P}_{\mathfrak{Q}}$ of "testable properties" (the image of $\sim$), while mapping each morphism to its restriction to testable properties. We provide precise definitions as follows.

The functor $\mathbf{F} : \mathbb{L} \to \mathbb{Q}$ maps an orthomodular lattice $\mathfrak{L} = (L, \leq, -^{\perp})$ to the quantum dynamic algebra consisting of sets of functions on L that are compositions of functions f_p for $p \in L$. Let $\mathscr{F}_T$ be the smallest set containing $\{f_p \mid p \in L\}$ and closed under function composition $\circ$; recall that f_p is the Sasaki projection onto p. Define $\mathscr{Q} \stackrel{\text{def}}{=} \wp(\mathscr{F}_T)$ and let

$$\cdot : \mathscr{Q} \times \mathscr{Q} \to \mathscr{Q} :: A \cdot B \mapsto \{a \circ b \in \mathscr{F}_T \mid a \in A \, and \, b \in B\}, \text{ and } \sim : \mathscr{Q} \to \mathscr{Q} :: A \mapsto \{f_{(\bigvee\{a(I) \mid a \in A\})^{\perp}}\}.$$

It is easy to see that the tuple $\mathbf{F}(\mathfrak{L}) = (\mathscr{Q}, \bigcup, \cdot, \sim)$ is a generalized dynamic algebra. We show that $\mathbf{F}(\mathfrak{L})$ is an orthomodular dynamic algebra by verifying the conditions (4)–(10) in the definition. The functor $\mathbf{F}$ maps each $\mathbb{L}$-morphism k from $\mathfrak{L}_1 = (L_1, \leq_1, -^{\perp_1})$ to $\mathfrak{L}_2 = (L_2, \leq_2, -^{\perp_2})$ to a mapping $\mathbf{F}(k) : \mathscr{Q}_1 \to \mathscr{Q}_2$ that conjugates each element of the input set: i.e. $\mathbf{F}(k) :: A_1 \mapsto \{k \circ a \circ k^{-1} \mid a_1 \in A_1\}$. We show that $\mathbf{F}(k)$ is a $\mathbb{Q}$-morphism from $\mathbf{F}(\mathfrak{L}_1)$ to $\mathbf{F}(\mathfrak{L}_2)$ and establish $\mathbf{F}$ as a functor from $\mathbb{L}$ to $\mathbb{Q}$.

In the other direction $\mathbf{U} : \mathbb{Q} \to \mathbb{L}$ maps a quantum dynamic algebra $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$ to $(\mathscr{P}, \preceq, \sim)$ with $\mathscr{P} = \{\sim x \mid x \in Q\}$, and we show that $(\mathscr{P}, \preceq, \sim)$ is indeed a complete orthomodular lattice. For each $\mathbb{Q}$-morphism $\theta : \mathfrak{Q}_1 \to \mathfrak{Q}_2$ let $\mathbf{U}(\theta)$ be the restriction of θ to $\mathscr{P}_1$. We show that $\mathbf{U}(\theta)$ is an $\mathbb{L}$-morphism from $\mathbf{U}(\mathfrak{Q}_1)$ to $\mathbf{U}(\mathfrak{Q}_2)$ and hence establish $\mathbf{U}$ as a functor from $\mathbb{Q}$ to $\mathbb{L}$.

For each complete orthomodular lattice $\mathfrak{L} = (L, \leq, -^{\perp})$, we let $\tau_{\mathfrak{L}}(p) = \{f_p\}$ and show that $\tau : 1_{\mathbb{L}} \to \mathbf{U} \circ \mathbf{F}$ is a natural isomorphism. Similarly, for each orthomodular dynamic algebra $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$, we define a function $\eta_{\mathfrak{Q}} : 1_{\mathbb{Q}}(\mathfrak{Q}) \to (\mathbf{F} \circ \mathbf{U})(\mathfrak{Q})$ as $\eta_{\mathfrak{Q}} :: x = \bigsqcup\{p_{(i,1)} \cdot \dots \cdot p_{(i,n_i)} \mid i \in I\} \mapsto \{f_{p_{(i,1)}} \circ \dots \circ f_{p_{(i,n_i)}} \mid i \in I\}$ where $p_{(i,j)} \in \mathscr{P}_{\mathfrak{Q}}$, for each $(i, j) \in \bigcup_{i \in I}(\{i\} \times \{1, \dots, n_i\})$. We show that for each $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$, $\eta_{\mathfrak{Q}}$ is an isomorphism in $\mathbb{Q}$ and η satisfies naturality and hence $\eta : 1_{\mathbb{Q}} \to \mathbf{F} \circ \mathbf{U}$ is a natural isomorphism. Finally, we come to the conclusion of this paper.

Theorem 1 $(\mathbf{F}, \mathbf{U}, \tau, \eta)$ *forms a categorical equivalence between* $\mathbb{L}$ *and* $\mathbb{Q}$.

Remark 3.1 *Had we replaced* complete orthomodular lattice *with a more restricted type of lattice such as a Hilbert lattice or Piron lattice, we could then show these quantum dynamic algebras are categorically equivalent to the corresponding category of latices (Hillbert lattice or Piron lattice with ortholattice isomorphisms) using the same functors and natural isomorphisms as we use here. The proof remains exactly the same. This thus establishes a link between quantum dynamic algebras and Hilbert spaces (or other important quantum structures) via these stronger lattice structures, particularly the Hilbert lattices.*

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