Open Petri Nets [1]

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Petri nets are a simple and widely studied model of computation, with generalizations applicable to many forms of modeling.

Definition 1. We define a **Petri net** to be a pair of functions of the following form:

$$T \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{N}[S]$$

We call T the set of **transitions**, S the set of **places**, s the **source** function and t the **target** function. A morphism of Petri nets is a pair of functions between the places and transitions of the respective Petri nets which commutes suitably with the source and target functions. This defines a category Petri of Petri nets.

Recently more attention has been paid to a compositional treatment in which Petri nets can be assembled from smaller 'open' Petri nets [2]. In particular, the reachability problem for Petri nets, which asks whether one marking of a Petri net can be obtained from another via a sequence of transitions, can be studied compositionally [3]. Here we seek to give this line of work a firmer footing in category theory.

Petri nets are closely tied to symmetric monoidal categories in two ways. First, a Petri net P can be seen as a presentation of a free symmetric monoidal category FP, with the places and transitions of P serving to freely generate the objects and morphisms of FP. In these terms, the reachability problem asks whether there is a morphism from one object of FP to another.

Second, there is a symmetric monoidal category where the objects are sets and the morphisms are equivalence classes of open Petri nets.

Definition 2. An **open Petri net** $P\colon X \nrightarrow Y$ is a cospan of Petri nets of the form

$$LX \xrightarrow{\;i\;} P \xleftarrow{\;o\;} LY$$

where $L\colon \mathsf{Set} \to \mathsf{Petri}$ is the functor which sends a set X to the Petri net which has X as its set of species and no transitions.

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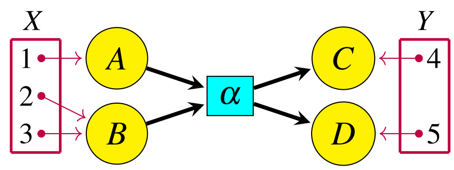
The open Petri net $P\colon X \nrightarrow Y$ can be represented with the following picture:



The yellow circles are places and the blue rectangle is a transition. The bold arrows from places to transitions and from transitions to places complete the structure of a Petri net. There are also arbitrary functions from X and Y into the set of places. These indicate points at which tokens could flow in or out, making our Petri net ‘open’. Open Petri nets can be composed by gluing the outputs of one to the inputs of another.

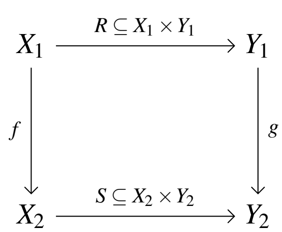
Instead of constructing a symmetric monoidal category of open Petri nets, we build a symmetric monoidal double category, $\mathbb{O}\mathbf{pen}(\mathsf{Petri})$, of open Petri nets. This decategorifies into a symmetric monoidal category by taking equivalence classes of 1-cells. $\mathbb{O}\mathbf{pen}(\mathsf{Petri})$ has

- sets $X, Y, Z, \ldots$ as objects,
- functions $f\colon X \to Y$ as vertical 1-morphisms,
- open Petri nets $P\colon X \nrightarrow Y$ as horizontal 1-cells,
- morphisms between open Petri nets as 2-morphisms, i.e. morphisms of Petri nets between the legs and apex of the cospans commuting appropriately.

The 2-cells in $\mathbb{O}\mathbf{pen}(\mathsf{Petri})$ allow for course-graining of systems; larger, more complicated open Petri nets can be simulated by smaller ones using these 2-cells.

The main goal of this paper is to describe the reachability semantics for open Petri nets as a map from $\mathbb{O}\mathbf{pen}(\mathsf{Petri})$ to the double category of relations, $\mathbb{R}\mathbf{el}$, which has:

- sets $X, Y, Z, \ldots$ as objects,
- functions $f\colon X \to Y$ as vertical 1-morphisms,
- relations $R \subseteq X \times Y$ as horizontal 1-cells,
- squares



obeying $(f \times g)R \subseteq S$ as 2-morphisms.

In Petri net theory, a 'marking' of a set X is a finite multisubset of X: we can think of this as a way of placing finitely many tokens on the points of X. Let $\mathbb{N}[X]$ denote the set of markings of X. Given an open Petri net $P\colon X \nrightarrow Y$, there is a 'reachability relation' saying when a given marking of X can be carried by a sequence of transitions in P to a given marking of Y, leaving no tokens behind. We write the reachability relation of P as

$$\blacksquare P \subseteq \mathbb{N}[X] \times \mathbb{N}[Y].$$

The following theorem gives a categorical and commpositional description of reachability for open Petri nets.

Theorem 3. *There is a lax symmetric monoidal double functor* $\blacksquare\colon \mathbb{O}\mathbf{pen}(\mathsf{Petri}) \to \mathbb{R}\mathbf{el}$*, called the* **reachability semantics***, that sends*

- *any object* X *to the underlying set of the free commutative monoid* $\mathbb{N}[X]$*, which we denote simply as* $\mathbb{N}[X]$*,*
- *any vertical 1-morphism* $f\colon X \to Y$ *to the underlying function of* $\mathbb{N}[f]$*,*
- *any horizontal 1-cell, that is, any open Petri net*

$$LX \xrightarrow{i} P \longleftarrow LY,_o$$

to the reachability relation $\blacksquare P$*.*

- *any 2-morphism* $\alpha\colon P \Rightarrow P'$ *which consists of a tuple* $(f\colon X \to X', h\colon P \to P', g\colon Y \to Y')$ *is sent to the inclusion of relations*

$$(\mathbb{N}[f] \times \mathbb{N}[g])\blacksquare P \subseteq \blacksquare P'.$$

We show that this double functor only preserves composition of open Petri nets up to a 2-cell in $\mathbb{R}\mathbf{el}$; the reachability relation of the composite of two open Petri nets can strictly contain the composite of their reachability relations. This upper bound on compositional reachability gives a method for designing systems using Petri nets in a compositional and formally verified way.

References

[1] J. C. Baez and J. Master, Open Petri Nets. Available at https://arxiv.org/abs/1808.05415

[2] P. Baldan, A. Corradini, H. Ehrig, and R. Heckel, Compositional semantics for open Petri nets based on deterministic processes, *Math. Str. Comp. Sci.* **15** (2005), 1–35.

[3] J. Rathke, P. Sobociński and O. Stephens, Compositional reachability in Petri nets, in *International Workshop on Reachability Problems*, Lecture Notes in Computer Science **8762**, Springer, Berlin, 2014. Available at http://users.ecs.soton.ac.uk/ps/papers/rp2014.pdf.