Homotopical approach to magic assignments

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Abstract

We extend Arkhipov's graph theoretic characterization of magic assignments to arbitrary collection of contexts by relaxing the requirement that each observable belongs to exactly two contexts. We introduce topological realizations of operator constraints by 2-dimensional cell complexes. Then we prove that the assignment is non-magic if the first homotopy group of the topological realization is trivial.

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1.1 Introduction

Quantum assignments that are not classically realizable can be seen as proofs of contextuality generalizing the Mermin-Peres square [4]. Arkhipov [1] studied magic assignments under two restrictions (1) eigenvalues of the observables are ± 1, and (2) each observable belongs to exactly two contexts. In this work we extend his results to magic assignments without these restrictions by using the topology of 2-dimensional cell complexes.

A topological framework for contextuality is introduced in [5]. Interpretation of contexts as triangulation of surfaces turns out to be a very fruitful approach that extracts the topological basis in contextuality proofs. To study magic assignments we switch from triangulations to cell complexes. These topological constructions are more flexible and suitable for studying operator constraints. A classical solution to such constraints can be interpreted as a cohomology class, and the first homotopy group of the cell complex determines whether the cohomology class vanishes allowing a classical realization.

1.2 Magic assignments

We briefly recall the relevant definitions. Let L be a finite set of labels for the observables, $\mathcal{M}$ be a collection of subsets. The members of $\mathcal{M}$ are referred to as contexts. The operators will be denoted by T_a where a is a label in L. A *quantum realization* consists of operators satisfying the following properties:

- Each operator satisfies $(T_a)^d = I$.
- For each context the operators $\{T_a |\ a \in C\}$ pairwise commute.
- For each context $C \in \mathcal{M}$ the operators satisfy the constraint

$$\prod_{a\in C} T_a^{\epsilon(a)} = \omega^{\tau(C)} \tag{1}$$

where ω denotes the d-th root of unity $e^{2\pi i/d}$ for some $d > 0$. Therein $\epsilon(a) = \pm 1$, that is we allow operator inverses to appear in the constraints. This is an extra feature that results from considering arbitrary d.

A *classical realization* consists of scalars $\omega^{c(a)}$ for each label such that for each context they satisfy

$$\prod_{a\in C} \omega^{\epsilon(a)c(a)} = \omega^{\tau(C)}. \tag{2}$$

We say an arrangement is *(classically) quantum realizable* if it has a (classical) quantum realization for some τ. An arrangement is called *magic* if it is quantum realizable but not classically realizable.

1.3 Main results

Under the conditions that (1) $T_a^2 = I$ and (2) each a belongs to exactly two contexts Arkhipov [1] proves that an arrangement is magic if and only if the intersection graph is not planar. Here the intersection graph is obtained from $(L, \mathcal{M})$ by putting an edge between two labels whenever they belong to the same context.

In this work we remove these two conditions. Namely, our operators satisfy $(T_a)^d = I$ for an arbitrary $d \geq 2$, and there is no restriction on the way the contexts intersect. Now within this generality the role of the intersection graph is replaced by a 2-dimensional space that encodes the operator constraints. More precisely, we define a notion called *topological realization* (see §1.4) that is a space realizing a set of constraints in a topological way. The topological realization is a 2-dimensional cell complex X whose edges are labeled by L and its faces are labeled by contexts. In our framework the planarity of the intersection graph is replaced by a topological condition on X. A space is called simply connected if the first homotopy group $\pi_1(X)$, also known as the fundamental group, is trivial. In other words, in such a space every closed loop is contractible.

Theorem 1.1. *If an arrangement $(L, \mathcal{M})$ is topologically realizable by a simply connected space X then it is non-magic.*

We refine this result further by describing the additional structure of operator constraints as a projective representation of the fundamental group. Given a closed loop p on X that consists of a sequence of edges we can consider the loop operator T_p that is the product of the operators that label the edges in this sequence of edges. Our main technical result (Lemma 1.4) says that if q is another loop homotopic to p the loop operator T_q differs from T_p by a scalar, in fact by a power of ω. Thus we replace the equivalence relation of homotopy by the equivalence relation of multiplying by a scalar. This construction gives us a projective representation

$$T : \pi_1(X) \to PU(n)$$

where $PU(n)$ is the projective unitary group. Sometimes we can replace each loop operator by a scalar and obtain a linear representation, that is lifting the representation to the unitary group $U(n)$. We show that this linearization operation can be performed if and only if there is a classical realization (Theorem 1.6). Then studying the linearization problem we are able to extend Theorem 1.1.

Theorem 1.2. *If an arrangement has a topological realization X where $\pi_1(X)$ is a finite group whose order is coprime to d then the arrangement is non-magic.*

Example 1.3. Let us look at the Mermin square example as described in [5]. The operators labels the edges of the surface and faces correspond to contexts. There are two ways to orient the edges as in Fig. (1). After identifying the outer edges the left figure, which we will denote by X^+, gives a torus. The other orientation gives a space, denoted by X^-, which is quite distinct. This choice of orientation gives the projective plane. Note that

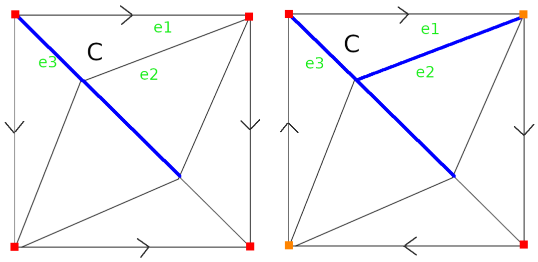

Figure 1: Two different topological realizations for the Mermin square.

X^+ has a single vertex (0-cell), whereas X^- has two vertices. The edges and faces in both cases are the same.
Let us demonstrate Theorem 1.2 in this example. The first homotopy groups are given by

$$\pi_1(X^+) = \mathbb{Z} \times \mathbb{Z} \quad \text{and} \quad \pi_1(X^-) = \mathbb{Z}_2.$$

In the latter case, the fundamental group is finite of order 2. Thus Theorem 1.2 implies that for d odd any labeling of the edges of X^- by operators T_a satisfying $(T_a)^d = I$ is classically realizable.

1.4 Topological realizations

Our main technical tool to study arbitrary arrangements is a topological construction known as a cell complex [3]. Cell complexes are constructed by gluing elementary pieces together to form a space. This construction is a generalization of the triangulation approach taken in [5]. More precisely the procedure is as follows

- Start with a set X^0 of 0-cells;
- Construct the n-the skeleton X^n by attaching n-cells to the $(n-1)$-st skeleton X^{n-1}.

An n-cell is a disk D^n of dimension n. Note that its boundary is the n-sphere S^n. Each cell comes with a characteristic map $\Phi_j : D^n \to X$. Restriction of this map to the boundary ∂D^n gives us the attaching map.

Given an arrangement $(L, \mathcal{M})$ together with the operator constraints in Eq. (1) we introduce a cell complex that encodes these constraints in the form of attaching maps. A *topological realization* is a cell complex X whose 1-cells are labeled by L, and its 2-cells are labeled by contexts in $\mathcal{M}$ such that these 2-cells are glued to the 1-skeleton X^1 by traversing the labels

$a \in C$ with the orientation $\epsilon(a)$ as specified in Eq. (1). Thus the operator constraints are encoded in the cell structure. It is important to note that the set of 0-cells is arbitrary, except that we require the resulting space to be connected. For example, in Fiq. (1) we see two topological realization with different number of vertices.

Using this definition we can express Eq. (1) in a more topological way. By construction each 2-cell with characteristic map $\Phi_j : D^2 \to X$ knows about the corresponding operator constraint for the context C_j. Eq. (1) contains an extra bit of information, namely the function $\tau : \mathcal{M} \to \mathbb{Z}_d$. We interpret this function as a 2-cochain on X since it can be thought of as a function on the 2-cells. We rewrite Eq. (1) using the path operator notation

$$T_{\Phi_j(\partial D^2)} = \omega^{\Phi_j^* \tau(D^2)}. \tag{3}$$

Therein, $T_{\Phi_j(\partial D^2)}$ is the path operator corresponding to the loop $\Phi_j(\partial D^2)$, and $\Phi_j^* \tau(D^2)$ means that we evaluate the 2-cocycle τ on the 2-cell labeled by C_j.

The key observation in the proof of Theorem 1.1 is a generalization of Eq. (3). At this point we need to work with a more rigid version of cell complexes. In fact our topological realization space is a special type of cell complex called a combinatorial complex [2]. The suitable notion of a map between two combinatorial complexes is a combinatorial map. Such maps have very useful properties that allows us to generalize Eq. (3) to the following key result which plays an important role in the proof of Theorem 1.1.

Lemma 1.4. *For a combinatorial map $g : D^2 \to X$ we have*

$$T_{g(\partial D^2)} = \omega^{g^* \tau(D^2)}.$$

Let us illustrate the importance of this result. Consider the 2-sphere S^2. A continuous map $h : S^2 \to X$ can be decomposed into two maps $g_1 : D^2 \to X$ and $g_2 : D^2 \to X$ such that $g_1(\partial D^2) = g_2(\partial D^2)$. Here we think of S^2 as composed of two hemispheres. We can evaluate τ on the sphere by evaluating it on each hemisphere. Applying Lemma 1.4 to g_1 implies that evaluation on a hemisphere is given by the loop operator T_p where p is the loop given by the image of the equator under h. Evaluation on the other hemisphere corresponding to g_2 will give T_{-p} where $-p$ is the original loop p with reversed orientation. These two contributions cancel each other to give

$$\omega^{h^* \tau(S^2)} = T_p T_{-p} = I.$$

That is evaluation on the sphere is always gives the identity operator. This reflects the fact that any loop on a sphere is contractible. If we were to replace the sphere with a higher genus surface, such as the torus X^+ in Fig. (1), we would end up with a non-identity element reflecting the non-trivial fundamental group of the surface. Indeed, for the Mermin's contradiction ($d = 2$) the outer edges of X^+ are labeled by anti-commuting operators which implies that the evaluation on X^+ is $-I$.

1.5 Holonomy representation

By studying the topology of 2-dimensional cell complexes we will interpret Lemma 1.4 as a representation of the fundamental group $\pi_1(X)$. A homotopy between two paths is essentially a map from the disk. Our key lemma says that there will be a phase associated to this homotopy. Therefore the resulting representation is in fact a projective representation i.e. only well-defined up to a phase.

The construction of the representation relies on the fundamental sequence [2] of a 2-dimensional cell complex. This is an exact sequence consisting of homotopy groups

$$0 \to \pi_2(X) \to \pi_2(X, X^1) \xrightarrow{\partial} \pi_1(X^1) \to \pi_1(X) \to 1. \tag{4}$$

Let us describe the groups in this sequence:

- $\pi_1(X)$ is the fundamental group of X consisting of homotopy classes of loops $S^1 \to X$ based at a fixed vertex v.
- Second homotopy group $\pi_2(X)$ is the homotopy classes of maps $S^2 \to X$ based at v.
- The relative homotopy group $\pi_2(X, X^1)$ consists of homotopy classes of maps $g : D^2 \to X$ such that $g(\partial D^2)$ is a loop in X^1 based at v.

The 1-skeleton X^1 is an oriented graph and it is easy to understand the non-contractible loops in it. Let $a_1, \cdots, a_n$ denote these non-contractible loops. There are two topological facts here

- $\pi_1(X^1)$ is generated (freely) by the collection of loops $a_1, \cdots a_n$;
- $\pi_2(X, X^1)$ is generated by the characteristic maps Φ_j of the 2-cells corresponding to each context.

Now the boundary map ∂ in the fundamental sequence Eq. (4) gives an interesting description of the operator constraints in Eq. (3). A characteristic map Φ_j will be sent under the boundary map ∂ to a word $a_{i_1}^{\epsilon^1} \cdots a_{i_r}^{\epsilon^r}$ in the free group $\pi_1(X^1) = F(a_1, \cdots, a_n)$. Therefore the operator constraint for a context C_j turn into a loop operator constraint

$$\omega^{\Phi_j^* \tau(D^2)} = \prod_{l=1}^{r} T_{a_{i_l}}^{\epsilon_l}. \tag{5}$$

Note that although the left-hand side of this equation remains the same in comparison to Eq. (3), the loop operators on the right-hand side no longer commutes. We pause to illustrate the loop constraints on the example given in Fig. 1.

Example 1.5. Let $\mathcal{T}^+$ and $\mathcal{T}^-$ denote the 1-dimensional subcopmlexes denoted by the blue edges in Fig. (1). This is a special subcomplex with the property that adding one more edge would create self-loops. Therefore $\mathcal{T}^\pm$ is a choice of a maximal tree inside the 1-skeleton. Now for each edge that lies outside of $\mathcal{T}^\pm$ there will be a a_i loop generator. Consider the top context labeled by C. In both cases the operator constraint is

$$T_{e_1} T_{e_2} T_{e_3} = \omega^{\tau(C)}.$$

For X^+ there are two loops $a_1^+ = e_1$ and $a_2^+ = e_2 \cdot e_3$, whereas for X^- there is only a single loop $a_1^- = e_1 \cdot e_2 \cdot e_3$. So the corresponding loop constraints are

$$T_{a_1^+} T_{a_2^+} = \omega^{\tau(C)} \quad \text{and} \quad T_{a_1^-} = \omega^{\tau(C)}.$$

In fact, writing out all the loop constraints one can verify that X^- always has a classical realization when d is odd. Therefore the loop constrains can be thought of a better, more useful way of reorganizing the original constraints.

An analysis of the loop constrains in Eq. (5) gives a representation theoretic characterization of non-magic assignments. Theorem 1.2 is almost an immediate consequence of the following characterization.

Theorem 1.6. *A quantum realization for an arrangement induces a projective representation $T : \pi_1(X) \to U(n)/\langle \omega \rangle$ such that the following conditions are equivalent*

- *The arrangement is non-magic.*
- *T can be lifted to a linear representation.*

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