Quantum accuracy enhancing protocols for clocks [1]
(Abstract-only submission to QPL19)

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I. INTRODUCTION

The steady development of quantum technologies over the past few years has also had an impact on timekeeping. Indeed, today's most accurate clocks are sophisticated quantum devices [2–4]. There are also fundamental reasons suggesting that it is favorable to exploit quantum phenomena when building clocks. For example, in a recent work by Woods, Pütz, Stupar, and two of the present authors [5], it was shown that clocks built with quantum systems can be quadratically more accurate than clocks built with classical systems of the same dimension.

However, this result concerns single clocks, which are isolated from each other. Here we ask whether interaction between such clocks can provide an enhancement of the time information produced by them. An answer to this question would inspire new mechanisms of more reliable time-tracking and with that contribute to research on the frontier of physics.

Concretely, we consider a setting where an internal clock receives a signal as input, possibly generated by another clock (denoted as the input clock), and produces a clock signal that is more accurate than both the input signal and the internal signal, as illustrated in Fig. 1. Simple as it is, this setting captures the essence of enhancing clocks and can be readily generalized to more complex settings. For instance, atomic clocks [3, 4, 6] are based on a similar idea of having a clock input its ticks into a congregation of atoms. Radio clocks also have the same structure where local clocks enhance their accuracy by receiving broadcast clock signals. We propose protocols where an internal system of dimension $d < \infty$ enhances the accuracy of an input clock signal. We show that, with a d-dimensional quantum system, it is possible to improve the accuracy of the input signal up to a factor of d (see Section III), outperforming other protocols that do not employ quantumness (see Section IV). Our protocols can be applied to designing highly accurate clocks and to establishing synchronization between clocks in a network.

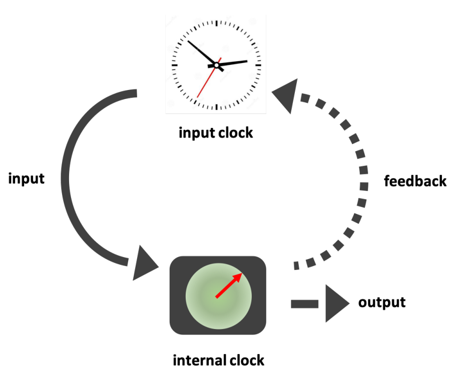

FIG. 1. **The accuracy enhancing task.** An internal clock system with fixed dimension receives a clock signal from an input clock and produces an enhanced clock signal as output. The performance of the protocol can be further improved when feedback on the input clock is allowed.

II. ACCURACY MEASURE

We model clocks as devices that output discrete sequences of ticks. (Note that in this work we consider the fundamental task of producing clock ticks, which should not be confused with the task of clock synchronization as is often considered in quantum metrology [7–9].) An accurate clock is expected to produce highly regular ticks. This regularity can be captured by the clock's ability to time events, i. e. we can assign the time tag "j" to an event if it happens after the j-th tick of the clock and before the $(j+1)$-th tick of the clock. A clock is more accurate if it has higher chance to time-tag an event unambiguously. Given a tail probability threshold ϵ, it is possible to find a narrow interval on the timeline such that we know, with confidence $1-\epsilon$, that a certain clock tick arrives within the interval. Then, whether a random event's time tag is ambiguous depends on whether it falls outside or within those intervals: If the event happens outside the intervals, then its time tag is unambiguous, but not if it happens inside an interval. The chance of an event being ambiguous is thus the ratio of the width of an interval to the distance between the intervals, and thus we can choose the inverse of this ratio as an accuracy measure. Precisely, we have the following definition of a clock's accuracy:

Definition 1 (Accuracy of a clock)**.** *Denoting by T_j the time between the first tick and the $(j+1)$-th tick,* a confidence interval *with tail probability ϵ is defined as an interval $\mathsf{C}_{j,\epsilon} = (\mu_{j,\epsilon} - \Sigma_{j,\epsilon}/2, \mu_{j,\epsilon} + \Sigma_{j,\epsilon}/2)$ that satisfies*

$$\Pr\left[T_j \notin \mathsf{C}_{j,\epsilon}\right] \leq \epsilon. \tag{1}$$

For a given tail probability ϵ, the accuracy of a clock signal is characterized by the set of quantities $R_{j,\epsilon}$, defined as

$$R_{j,\epsilon} := \sup_{\mathsf{C}_{j,\epsilon}} R\left(\mathsf{C}_{j,\epsilon}\right) \qquad R\left(\mathsf{C}_{j,\epsilon}\right) := \frac{\mu_{j,\epsilon}}{\Sigma_{j,\epsilon}} \tag{2}$$

for every j and ϵ, where $\mu_{j,\epsilon}$ and $\Sigma_{j,\epsilon}$ are defined with respect to the confidence interval $\mathsf{C}_{j,\epsilon}$.

III. ACCURACY ENHANCING PROTOCOL

Given an input clock signal, the goal of accuracy enhancement is to produce an output signal with as high accuracy as possible, using an internal clock as in Fig. 1. To model the clock, we extend the concept of autonomous clocks introduced in [10, 11] and developed in [12, 13], which produce signals without an external time reference. An autonomous clock is characterized by two key ingredients: a clock state (which can be either classical or quantum) and a detector which is a time-independent measurement that *constantly* measures the clock state and produces ticks [5, 11].

The most accurate autonomous clocks so far [14] are the *Quasi-ideal clocks* [13],[5]. Specifically, a Quasi-ideal clock of dimension d achieves an accuracy of $d^{1-\nu}$ for any positive ν and for any tail probability $\epsilon > 0$ in the limit of large d. As in Ref. [5], we take the clock to be a *reset* clock, namely that, every time it ticks, the clock state is always reset to the same quantum state, that we refer to in the following as its *reset state*.

To utilize the input signal in our accuracy enhancing protocol, we extend autonomous clocks by adding a switch that controls the detector of the clock. Dependent on the status of the switch, the clock system is governed by one of two different dynamics: $\mathcal{M}_{\text{on}}$, corresponding to the case when the detector is on and the clock is producing ticks, and $\mathcal{M}_{\text{off}}$ corresponding to the case when the detector is off and the clock evolves without being measured. In particular, when the clock is quantum, $\mathcal{M}_{\text{off}}$ corresponds to unitary dynamics that drives the clock to evolve periodically without any dissipation. Denote by d the dimension of the clock. To enhance the input signal, we require a quantum clock that becomes more accurate as d grows large and also is resilient to fluctuations in the switching between the two dynamics. This is made explicit by the following *clock condition*:

Definition 2 (The clock condition). *Consider a quantum clock initialized in the state Ψ_{t^*} ($t^* \in (-\tau^{(c)}/2, \tau^{(c)}/2]$) with a unitary evolution period of $\tau^{(c)}$ (i.e. the clock state remains the same after it evolves under $\mathcal{M}_{\text{off}}$ for $\tau^{(c)}$). Denote by $T^{(c)}$ the time it takes the clock to tick. We say such a quantum clock satisfies the clock condition if there exist two functions $\Sigma_d^{(c)}$ and $\epsilon_d^{(c)}$ of d, both vanishing in the limit of $d \to \infty$, such that for any $t^* \in \left(-\frac{\tau^{(c)}-\Sigma_d^{(c)}}{2}, \frac{\tau^{(c)}-\Sigma_d^{(c)}}{2}\right)$, the following inequality holds*

$$\Pr\left[T^{(c)} + t^* \in \left(\frac{\tau^{(c)} - \Sigma_d^{(c)}}{2}, \frac{\tau^{(c)} + \Sigma_d^{(c)}}{2}\right)\right] \geq 1 - \epsilon_d^{(c)}. \tag{3}$$

One can imagine that a clock satisfying the above condition has a "hand" that evolves on a clock face, and Eq. (3) requires that the clock ticks if and only if its hand hits a certain position (i. e. the location of the detector) on the clock face regardless of the initial position of the hand.

From Eq. (3), we can see that, for any $\epsilon > 0$, a quantum clock satisfying the clock condition and initiated in the state Ψ_0 has at least an accuracy

$$R_d^{(c)} := \frac{\tau^{(c)}}{2\Sigma_d^{(c)}} \tag{4}$$

for large enough d. The clock condition is satisfied by the *Quasi-ideal clock* [5, 13] (see [1, Lemma 2] for details).

Now we are ready to introduce our protocol, which enhances the accuracy of input signals without using feedback on the input signal. The protocol requires an internal quantum clock satisfying the clock condition in Definition 2, whose status at certain time is specified by the tuple $(\Psi_{t^*}, \mathcal{M})$, where Ψ_{t^*} denotes the state of the clock with a hand parameter $t^* \in (-\tau^{(c)}/2, \tau^{(c)}/2]$ and $\mathcal{M} \in \{\mathcal{M}_{\text{on}}, \mathcal{M}_{\text{off}}\}$ is the dynamics of the switch-controlled clock.

Protocol 1 Accuracy enhancing protocol without feedback.

1: (Initialization) When receiving the first input tick, reset the quantum clock to $(\Psi_0, \mathcal{M}_{\text{on}})$;
2: when the quantum clock ticks, produce an output tick and set the quantum clock to $(\Psi_0, \mathcal{M}_{\text{off}})$;
3: when receiving an input tick, set the quantum clock's dynamic to $\mathcal{M}_{\text{on}}$, i.e. turn on its detector.

Our main result on the performance of Protocol 1 is summarized as the following theorem, whose proof can be found in [1]:

Theorem 1. *Consider an i.i.d. input signal with a given ϵ-tail confidence interval $\left(\mu_\epsilon^{(\text{in})} - \Sigma_\epsilon^{(\text{in})}/2, \mu_\epsilon^{(\text{in})} + \Sigma_\epsilon^{(\text{in})}/2\right)$ satisfying $\Sigma_\epsilon^{(\text{in})} < 2\mu_\epsilon^{(\text{in})}/3$. For large enough d, the accuracy of the $(j+1)$-th output tick of Protocol 1 satisfies*

$$R_{j,\epsilon_j}^{(\text{out})} \geq \frac{6}{5j} \cdot R_\epsilon^{(\text{in})} \cdot R_d^{(c)} \qquad \epsilon_j = j \cdot \epsilon_0 \tag{5}$$

for any $\epsilon_0 > \epsilon$ and $j < 2\mu_\epsilon^{(\text{in})}/(3\Sigma_\epsilon^{(\text{in})})$. Here $R_\epsilon^{(\text{in})} = \left(\mu_\epsilon^{(\text{in})}/\Sigma_\epsilon^{(\text{in})}\right)^2$ is the input accuracy and $R_d^{(c)}$ is the accuracy of the internal quantum clock given by Eq. (4). When the quantum clock in Protocol 1 is taken to be a Quasi-ideal clock, the accuracy can be further bounded as

$$R_{j,\epsilon_j}^{(\text{out})} \geq \frac{3}{5j} \cdot R_\epsilon^{(\text{in})} \cdot d^{1-\nu} \tag{6}$$

for any $\nu > 0$.

The right hand side terms of Eqs. (5) and (6) drop as j grows, implying that the output signal becomes less accurate after the protocol runs for a long time. It turns out that this problem can be remedied when feedback on the input clock that produces the input ticks is allowed. More details of the feedback protocol can be found in [1].

IV. INCOHERENT PROTOCOLS

In view of the results above, it is natural to ask whether a similar improvement could be achieved by means of classical information processing, namely incoherent protocols that process the input ticks and/or the internal clock's ticks to produce a more accurate output signal. While we do not have a general bound, we present here two natural protocols and show that accuracy enhancement is possible, but with a lower factor. For simplicity, we omit tail probabilities in the following discussion, which can be regarded to be fixed.

First, we show a simple protocol using a d-dimensional classical memory that improves the input signal's accuracy:

Protocol 2 An incoherent protocol achieving accuracy enhancement by bunching input clock ticks.

(Initialization) Set the classical memory to the state $c = 0$;
when receiving an input tick:
if $c = d - 1$, **then** produce an output tick and reset the memory to the initial state $c = 0$;
else $c \to c + 1$.
end if

This classical memory can also be regarded as a clock driven by input signals. This protocol improves the accuracy of the input signal by *bunching* the input ticks. Notice that, due to bunching, the frequency of the input signal is also reduced by a factor of d.

In the following we propose another incoherent protocol, which is capable of preserving the frequency of the input signal. Imagine that we are given an internal clock system with much higher frequency than the input signal. This incoherent protocol works by ignoring all ticks of the internal clock but outputting only the one that follows immediately after every input tick, which can be done with the help of an additional classical bit c (This requirement does not affect our analysis, since we care only about the asymptotic scaling of the accuracies). Explicitly, the protocol works in the following manner:

Protocol 3 An incoherent protocol achieving accuracy enhancement by bunching internal clock ticks.

(Initialization) Set the classical memory to the state $c = 0$;
when receiving an input tick, set the classical memory to the state $c = 1$;
when the internal clock ticks:
if $c = 0$, **then** ignore the clock tick (and do nothing);
else ($c = 1$) produce an output tick and reset the memory to the state $c = 0$.
end if

Both incoherent protocols lose some accuracy compared to the coherent protocol, either by a factor of $\sqrt{R^{(\mathrm{c})}}$ or $\sqrt{R^{(\mathrm{in})}}$ (see [1] for the derivation). The performances of the incoherent protocols are compared to the coherent one in Table I. A numerical simulation, depicted in Fig. 2, shows that the protocol of bunching clock ticks, the protocol of bunching input ticks and the coherent protocol (introduced in Section III) are all capable of achieving an enhancement, while the performance of the coherent protocol is clearly above that of the other two protocols. Finally, we remark that the same idea works also in the setting where feedback is available.

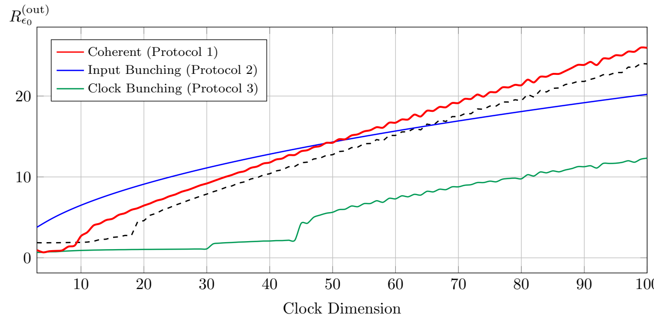

FIG. 2. **Numerical results for comparing the coherent protocol (Protocol 1), the input-bunching protocol (Protocol 2), and the clock-bunching protocol (Protocol 3).** The performances of different accuracy enhancing protocols on an i.i.d. box-shaped input clock of $R^{(\mathrm{in})}_{\epsilon} \approx 3$ with $\epsilon = 0.01$ are compared. The coherent protocol (red) and the input-bunching protocol (green) use a Quasi-ideal clock of dimension d whilst the input-bunching protocol (blue) uses a d-dimensional classical memory. The allowed error on the output is chosen to be the same as the one on the input (i. e. $\epsilon_0 = 0.01$). The black, dotted line corresponds to the asymptotic lower bound on the output accuracy in Eq. (5), plotted for a fixed clock error $\epsilon^{(\mathrm{c})} = 0.001$. The numerical simulation clearly shows an approximately linear scaling for both the coherent protocol and the clock-bunching protocol and $\sqrt{d}$ scaling for the input-bunching protocol. One can also see that the lower bound in Eq. (5) is quite tight even for low dimensions.

Protocols	Accuracy scaling
Coherent (Protocols 1)	$R^{(\mathrm{in})} \cdot R^{(\mathrm{c})}$
Bunching input clock ticks (Protocol 2)	$R^{(\mathrm{in})} \cdot \sqrt{R^{(\mathrm{c})}}$
Bunching internal clock ticks (Protocol 3)	$\sqrt{R^{(\mathrm{in})}} \cdot R^{(\mathrm{c})}$

TABLE I. Comparison between the asymptotic performances of coherent and incoherent protocols.

V. CONCLUSION

In this work, we proposed protocols that enhance the accuracy of a clock signal by a factor of order d, using a quantum system of dimension d. There are two key properties of the clock that lead to this accuracy enhancement. One is that, for the Quasi-ideal clock, its error is almost independent of its initial position, i.e. when the detector is switched on. The other is that, for most of the time in the protocol, the clock evolves without any dissipation. These two properties originate from the quantumness of the clock system. In this sense, our work justifies the power of quantumness in enhancing the accuracy of clocks. This point is further strengthened via the comparison with two incoherent protocols.

ACKNOWLEDGMENTS

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[14] i.e. those with analytical lower bounds on the accuracy.

Quantum accuracy enhancing protocols for clocks

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We study how interaction between clocks could enhance their production of time information. In particular, we consider a situation where a quantum clock receives a low-accuracy clock signal as input and ask whether it can generate an output of higher accuracy. We propose a protocol that, with a quantum clock of dimension d, achieves an accuracy enhancement by a factor d (in the limit of large d). If feedback on the input signal is allowed, our protocol is capable of retaining a high accuracy for a longer time. Our protocols can be applied to designing highly accurate clocks and to establishing synchronization between clocks in a network.

I. INTRODUCTION

The steady development of quantum technologies over the past few years has also had an impact on timekeeping. Indeed, today's most accurate clocks are sophisticated quantum devices [1–3]. There are also fundamental reasons suggesting that it is favorable to exploit quantum phenomena when building clocks. For example, in a recent work by Woods, Pütz, Stupar, and two of the present authors [4], it was shown that clocks built with quantum systems can be quadratically more accurate than clocks built with classical systems of the same dimension.

However, this result concerns single clocks, which are isolated from each other. Here we ask whether interaction between such clocks can provide an enhancement of the time information produced by them. An answer to this question would inspire new mechanisms of more reliable time-tracking and with that contribute to research on the frontier of physics.

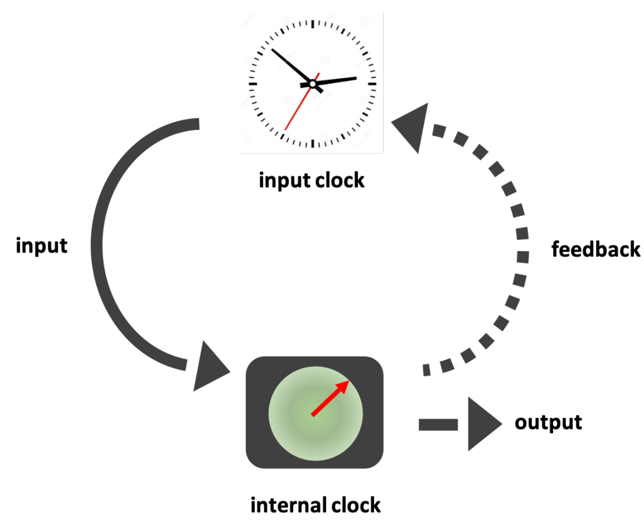

FIG. 1. **The accuracy enhancing task.** An internal clock system with fixed dimension receives a clock signal from an input clock and produces an enhanced clock signal as output. The performance of the protocol can be further improved when feedback on the input clock is allowed.

Concretely, we consider a setting where an internal clock receives a signal as input, possibly generated by another clock (denoted as the input clock), and produces a more accurate output signal than both the input signal and the internal signal, as illustrated in Fig. 1. Simple as it is, this setting captures the essence of enhancing clocks and can be readily generalized to more complex settings. For instance, atomic clocks [2, 3, 5] are based on a similar idea of having a clock input its ticks into a congregation of atoms. Radio clocks also have the same structure where local clocks enhance their accuracy by receiving broadcast clock signals. We propose protocols where an internal system of dimension $d < \infty$ enhances the accuracy of an input clock signal. We show that, with a d-dimensional quantum system, it is possible to improve the accuracy of the input signal up to a factor of d (see Section III), outperforming other protocols that do not employ quantumness (see Section V). If feedback on the input signal is allowed, our protocol is capable of becoming more accurate over time (see Section IV). Our protocols can be applied to designing highly accurate clocks and to establishing synchronization between clocks in a network (see Section VI).

II. ACCURACY MEASURE

We model clocks as devices that output discrete sequences of ticks. (Note that in this work we consider the fundamental task of producing clock ticks, which should not be confused with the task of clock synchronization as is often considered in quantum metrology [6–8].) An accurate clock is expected to produce highly regular ticks. This regularity can be captured by the clock's ability to time events, i. e. we can assign the time tag "j" to an event if it happens after the j-th tick of the clock and before the $(j+1)$-th tick of the clock. A clock is more accurate if it has higher chance to time-tag an event unambiguously. Given a tail probability threshold ϵ, it is possible to find a narrow interval on the timeline such that we know, with confidence $1-\epsilon$, that a certain clock tick arrives within the interval. Then, whether a random event's time tag is ambiguous depends on whether it falls outside or within those intervals: If the event happens outside the intervals, then its time tag is unambiguous, but not if it happens inside an interval. The chance of an event being ambiguous is thus the ratio of the width of an interval to the distance between the intervals, and thus we can choose the inverse of this ratio as an accuracy measure. Precisely, we have the following definition of a clock's accuracy:

Definition 1 (Accuracy of a clock)**.** *Denoting by T_j the time between the first tick and the $(j+1)$-th tick, a* confidence interval *with tail probability ϵ is defined as an interval $\mathsf{C}_{j,\epsilon} = (\mu_{j,\epsilon} - \Sigma_{j,\epsilon}/2, \mu_{j,\epsilon} + \Sigma_{j,\epsilon}/2)$ that satisfies*

$$\Pr\left[T_j \notin \mathsf{C}_{j,\epsilon}\right] \leq \epsilon. \tag{1}$$

For a given tail probability ϵ, the accuracy of a clock signal is characterized by the set of quantities $R_{j,\epsilon}$, defined as

$$R_{j,\epsilon} := \sup_{\mathsf{C}_{j,\epsilon}} R\left(\mathsf{C}_{j,\epsilon}\right) \qquad R\left(\mathsf{C}_{j,\epsilon}\right) := \frac{\mu_{j,\epsilon}}{\Sigma_{j,\epsilon}} \tag{2}$$

for every j and ϵ, where $\mu_{j,\epsilon}$ and $\Sigma_{j,\epsilon}$ are defined with respect to the confidence interval $\mathsf{C}_{j,\epsilon}$.

For i.i.d. clock signals, the accuracy of other ticks can be estimated from the accuracy of the first tick. Suppose $R_{1,\epsilon} = R(\mathsf{C}_{1,\epsilon})$ for some confidence interval $\mathsf{C}_{1,\epsilon} = (\mu_\epsilon - \Sigma_\epsilon/2, \mu_\epsilon + \Sigma_\epsilon/2)$. Applying Hoeffding's inequality within $\mathsf{C}_{1,\epsilon}$, we can bound the probability that the $(j+1)$-th tick falls out of a confidence interval of size $O(\sqrt{j}\cdot\Sigma_\epsilon)$. More precisely, we have

$$R_{j,\epsilon_{j,n}} \geq \frac{\sqrt{j}}{n} R_{1,\epsilon}, \tag{3}$$

where $\epsilon_{j,n} := 1-(1-\epsilon)^j(1-2e^{-n^2/2})$ and $n > 0$ is a parameter controlling the width of the confidence interval. While $R_{j,\epsilon_{j,n}}$ increases with j, the confidence $1-\epsilon_{j,n}$ decreases monotonically, which reflects the long-term effect of those undesired ticks on the accuracy. Such an effect can only be remedied when the tick's distribution has desired tail behavior. For instance, if the distribution of T_1 is sub-Gaussian (meaning that it has a tail that vanishes at least as fast as some Gaussian distribution), Hoeffding's inequality provides a tighter bound on the accuracy of the $(j+1)$-th tick as

$$R_{j,\epsilon} \geq f_\epsilon \cdot \sqrt{j} \cdot R_{1,\epsilon}, \tag{4}$$

where $f_\epsilon > 0$ depends only on ϵ and the distribution of T_1. Many types of clock signal distributions, including, for instance, Gaussian distributions, triangle-shaped distributions, and square-shaped distributions are sub-Gaussian. It also captures situations where the input is a mixture of several perfect signals (i. e. delta functions) with slightly different frequencies. In all these cases, Eq. (4) holds and the multiple tick accuracy increases linearly in the number of ticks considered.

Our accuracy measure can be compared to another measure of clock accuracy considered in [4, 9], which is defined as $\tilde{R}_j := (\mu_j/\sigma_j)^2$ with μ_j and σ_j being the mean and the standard deviation of T_j, respectively. Such a measure of accuracy is sensitive to the tail behavior of T_j's probability distribution, i. e. to ticks that deviate a lot from their means but happen with very small probability. For instance, if a clock produces ticks that deviate from a confidence On the other hand, for any clock tick achieving accuracy $\tilde{R}$ in terms of the measure considered in [4, 9], it is immediate from Chebyshev's inequality that it has accuracy

$$R_\epsilon \geq \sqrt{\epsilon \cdot \tilde{R}} \tag{5}$$

in terms of our measure. Since the other measure satisfies $\tilde{R}_j = j \cdot \tilde{R}_1$ for multiple i.i.d. ticks, we have

$$R_{j,\epsilon} \geq \sqrt{j \cdot \epsilon \cdot \tilde{R}_1} \tag{6}$$

for any clock producing i.i.d. signals.

III. ACCURACY ENHANCING PROTOCOL

Given an input clock signal, the goal of accuracy enhancement is to produce an output signal with as high accuracy as possible, using an internal clock as in Fig. 1. To model the clock, we extend the concept of autonomous clocks introduced in [10, 11] and developed in [9, 12], which produce signals without an external time reference. An autonomous clock is characterized by two key ingredients: a clock state (which can be either classical or quantum) and a detector which is a time-independent measurement that *constantly* measures the clock state and produces ticks [4, 11].

The most accurate autonomous clocks so far [13] are the *Quasi-ideal clocks* [12],[4]. Specifically, a Quasi-ideal clock of dimension d achieves an accuracy of $d^{1-\nu}$ for any positive ν and for any tail probability $\epsilon > 0$ in the limit of large d (see Ref. [4]; also see Appendix A for the proof in terms of the accuracy measure (2)). As in Ref. [4], we take the clock to be a *reset* clock, namely that, every time it ticks, the clock state is always reset to the same quantum state, that we refer to in the following as its *reset state*.

To utilize the input signal in our accuracy enhancing protocol, we extend autonomous clocks by adding a switch that controls the detector of the clock. Dependent on the status of the switch, the clock system is governed by one of two different dynamics: $\mathcal{M}_{\rm on}$, corresponding to the case when the detector is on and the clock is producing ticks, and $\mathcal{M}_{\rm off}$ corresponding to the case when the detector is off and the clock evolves without being measured. In particular, when the clock is quantum, $\mathcal{M}_{\rm off}$ corresponds to unitary dynamics that drives the clock to evolve periodically without any dissipation. Denote by d the dimension of the clock. To enhance the input signal, we require a quantum clock that becomes more accurate as d grows large and also is resilient to fluctuations in the switching between the two dynamics. This is made explicit by the following *clock condition*:

Definition 2 (The clock condition)**.** *Consider a quantum clock initialized in the state* Ψ_{t^*} $(t^* \in (-\tau^{(c)}/2, \tau^{(c)}/2])$ *with a unitary evolution period of* $\tau^{(c)}$ *(i.e. the clock state remains the same after it evolves under* $\mathcal{M}_{\rm off}$ *for* $\tau^{(c)}$*). Denote by* $T^{(c)}$ *the time it takes the clock to tick. We say such a quantum clock satisfies the clock condition if there exist two functions* $\Sigma_d^{(c)}$ *and* $\epsilon_d^{(c)}$ *of* d*, both vanishing in the limit of* $d \to \infty$*, such that for any* $t^* \in \left(-\frac{\tau^{(c)}-\Sigma_d^{(c)}}{2}, \frac{\tau^{(c)}-\Sigma_d^{(c)}}{2}\right)$*, the following inequality holds*

$$\Pr\left[T^{(c)} + t^* \in \left(\frac{\tau^{(c)} - \Sigma_d^{(c)}}{2}, \frac{\tau^{(c)} + \Sigma_d^{(c)}}{2}\right)\right] \geq 1 - \epsilon_d^{(c)}. \tag{7}$$

One can imagine that a clock satisfying the above condition has a "hand" that evolves on a clock face, and Eq. (7) requires that the clock ticks if and only if its hand hits a certain position (i. e. the location of the detector) on the clock face regardless of the initial position of the hand. Another key feature is that the width of the confidence interval remains unchanged, for different initial clock states.

From Eq. (7), we can see that, for any $\epsilon > 0$, a quantum clock satisfying the clock condition and initiated in the state Ψ_0 has at least an accuracy

$$R_d^{(c)} := \frac{\tau^{(c)}}{2\Sigma_d^{(c)}} \tag{8}$$

for large enough d. The clock condition is satisfied by the *Quasi-ideal clock* [4, 12] (see Lemma 2 in the Appendix for details).

Now we are ready to introduce our first protocol, which enhances the accuracy of input signals without using feedback on the input signal. The protocol requires an internal quantum clock satisfying the clock condition in Definition 2, whose status at certain time is specified by the tuple $(\Psi_{t^*}, \mathcal{M})$, where Ψ_{t^*} denotes the state of the clock with a hand parameter $t^* \in (-\tau^{(c)}/2, \tau^{(c)}/2]$ and $\mathcal{M} \in \{\mathcal{M}_{\rm on}, \mathcal{M}_{\rm off}\}$ is the dynamics of the switch-controlled clock. The specific steps of the accuracy enhancing protocol are as follows:

Protocol 1 Accuracy enhancing protocol without feedback.

1: (Initialization) When receiving the first input tick, reset the quantum clock to $(\Psi_0, \mathcal{M}_{\rm on})$;
2: when the quantum clock ticks, produce an output tick and set the quantum clock to $(\Psi_0, \mathcal{M}_{\rm off})$;
3: when receiving an input tick, set the quantum clock's dynamic to $\mathcal{M}_{\rm on}$, i.e. turn on its detector.

Before stating our main result, we give an illustrative explanation of why this protocol achieves an accuracy enhancement. The central idea is that, for a tail probability ϵ_0, we would like the output ticks to have a confidence interval $\left(\mu_{\epsilon_0}^{(\rm out)} - \Sigma_{\epsilon_0}^{(\rm out)}/2, \mu_{\epsilon_0}^{(\rm out)} - \Sigma_{\epsilon_0}^{(\rm out)}/2\right)$ such that $\mu_{\epsilon_0}^{(\rm out)} \approx \mu_\epsilon^{(in)}$, but to have a confidence interval width which scales like that of the quantum clock, i.e. $\Sigma_{\epsilon_0}^{(\rm out)} \approx \Sigma_d^{(c)}$. Since, by Definition 2, $\Sigma_d^{(c)}$ is small for large d, one enhances the accuracy.

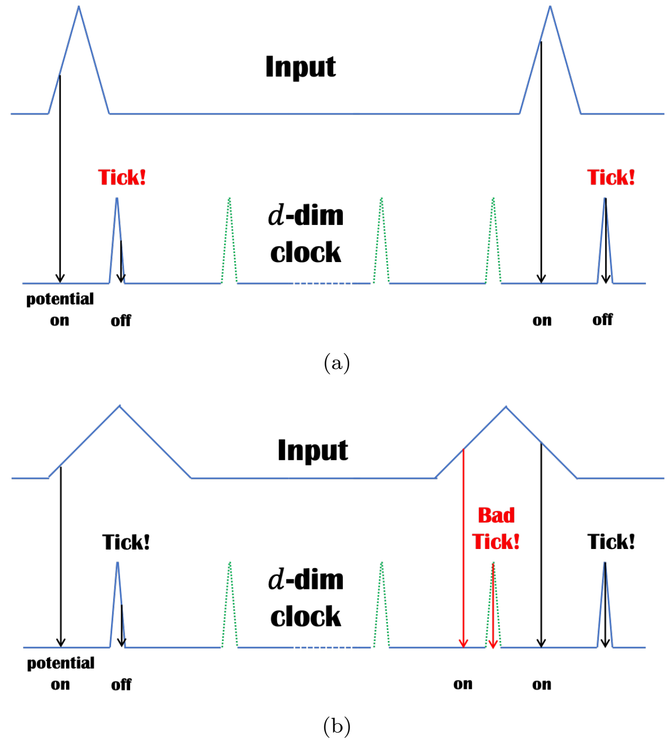

FIG. 2. **Tick patterns of the input signal and the clock in the accuracy enhancing protocol.** In Fig. 2(a), it is illustrated how output ticks are generated. Intervals where ticks are generated with high probability are depicted as pikes, and green, dotted pikes denote clock ticks that are suppressed when the clock evolves unitarily. Pikes in the upper half of the figure correspond to ticks of the input signal, which cause the potential of the clock to be turned on, whereas the output ticks are highlighted in red. In Fig. 2(b), $\Sigma_\epsilon^{(\rm in)}$ is wider than the period of the clock. In this case, the input tick may arrive too early, causing the output tick to be produced one period earlier than it should be (as highlighted in red in the figure).

For this purpose, we first select the period of the quantum clock to be much smaller than that of the input signal, so that an output tick is always generated shortly after every input tick and thus $\mu_{\epsilon_0}^{(\rm out)} \approx \mu_\epsilon^{(\rm in)}$. Next, by the clock condition, the quantum clock always ticks when the clock state hits the detector, regardless of the initial position of the clock state (see Definition 2). Thanks to this feature, input ticks that arrive within the same period of the quantum clock will trigger output ticks at almost the same time, up to a deviation that depends only on the quantum clock. Therefore, a fluctuation in the arrival of the input tick will not affect the output ticks, and $\Sigma_{\epsilon_0}^{(\rm out)} \approx \Sigma_d^{(c)}$ (see Fig. 2(a) for an illustration). It follows that the output accuracy can be expressed as $R_{\epsilon_0}^{(\rm out)} \approx \mu_\epsilon^{(\rm in)}/\Sigma_d^{(c)} = R_\epsilon^{(\rm in)} \cdot \left(\Sigma_\epsilon^{(\rm in)}/\Sigma_d^{(c)}\right)$.

However, the accuracy cannot be increased arbitrarily by this method. Since the input tick is required to fall within the same period of the Quasi-ideal clock, $\tau^{(c)}$ should not be smaller than $\Sigma_\epsilon^{(\text{in})}$. Otherwise, a bad tick with large deviation may be produced (see Fig. 2(b)). The probability for the output tick to be in its confidence interval is at least the product of the probabilities that the input tick and the clock tick to be in their own confidence intervals, respectively, which means $\epsilon_0 > \epsilon$. Then one finds that the best accuracy of our protocol's output signal is approximately the product of the accuracies of the input signal and clock system:

$$R_{\epsilon_0}^{(\text{out})} \approx R_\epsilon^{(\text{in})} \cdot R_d^{(\text{c})} \tag{9}$$

for $\epsilon_0 > \epsilon$ and for large enough d.

If we set the quantum clock to be the Quasi-ideal clock, which achieves an accuracy scaling as $d^{1-\nu}$ for any $\nu > 0$, it can be seen from the above equation that our protocol achieves an accuracy enhancement of almost d. In summary, we have the following theorem whose proof can be found in Appendix A:

Theorem 1. *Consider an i.i.d. input signal with a given ϵ-tail confidence interval $\left(\mu_\epsilon^{(\text{in})} - \Sigma_\epsilon^{(\text{in})}/2, \mu_\epsilon^{(\text{in})} + \Sigma_\epsilon^{(\text{in})}/2\right)$ satisfying $\Sigma_\epsilon^{(\text{in})} < 2\mu_\epsilon^{(\text{in})}/3$. For large enough d, the accuracy of the $(j+1)$-th output tick of Protocol 1 satisfies*

$$R_{j,\epsilon_j}^{(\text{out})} \geq \frac{6}{5j} \cdot R_\epsilon^{(\text{in})} \cdot R_d^{(\text{c})} \qquad \epsilon_j = j \cdot \epsilon_0 \tag{10}$$

for any $\epsilon_0 > \epsilon$ and $j < 2\mu_\epsilon^{(\text{in})}/(3\Sigma_\epsilon^{(\text{in})})$. Here $R_\epsilon^{(\text{in})} = \left(\mu_\epsilon^{(\text{in})}/\Sigma_\epsilon^{(\text{in})}\right)^2$ is the input accuracy and $R_d^{(\text{c})}$ is the accuracy of the internal quantum clock given by Eq. (8). When the quantum clock in Protocol 1 is taken to be a Quasi-ideal clock, the accuracy can be further bounded as

$$R_{j,\epsilon_j}^{(\text{out})} \geq \frac{3}{5j} \cdot R_\epsilon^{(\text{in})} \cdot d^{1-\nu} \tag{11}$$

for any $\nu > 0$.

The main result of this section, stated as Theorem 1, is that Protocol 1 can improve an input signal's accuracy almost by a factor equal to the quantum clock's dimension, given that the input signal is i.i.d.. On the other hand, the right hand side terms of Eqs. (10) and (11) drop as j grows, implying that the output signal becomes less accurate after the protocol runs for a long time. The tail probability grows linearly in j, which is compatible to the scaling of the input tick's tail probability before the accuracy enhancement as given by Eq. (3). Note that the period of the quantum clock in the protocol depends on the confidence interval of the input signal (see Appendix A for details).

IV. ACHIEVING LONG TERM STABILITY WITH FEEDBACK

A problem of the previous protocol is that the accuracy of ticks drops down after the protocol runs for a certain time. It turns out that this problem can be remedied when feedback on the input clock that produces the input ticks is allowed. The role of the feedback is to reset the input clock to its initial configuration in each round of producing a tick.

For any input signal generated by an input clock, the following protocol uses a quantum clock that satisfies the clock condition in Definition 2 and feedback on the input clock to produce output ticks:

Protocol 2 Accuracy enhancing protocol with feedback.

1: (Initialization) When receiving the first input tick, reset the quantum clock to $(\Psi_0, \mathcal{M}_{\text{on}})$;
2: when the input clock ticks, set the quantum clock's dynamic to $\mathcal{M}_{\text{on}}$, i.e. turn on its detector;
3: when the quantum clock ticks, produce an output tick, set the quantum clock to $(\Psi_0, \mathcal{M}_{\text{off}})$, and reset the input clock.

The working principle of this protocol is similar to that of the feedback-free protocol, and thus its performance is close to that of the feedback-free protocol in the short term. The long-term stability of the feedback protocol follows from the fact that it invokes a reset mechanism: When an output tick is produced, the input clock is reset to its initial configuration according to the last step of the protocol. At the same time, the quantum clock also ticks and thus resets, as mentioned in Section IV. Therefore, the output ticks are i.i.d., and the accuracy of the $(j+1)$-th output tick increases linearly in $\sqrt{j}$, as described by Eq. (3). To summarize, the performance of the feedback protocol is as good as the feedback-free protocol and becomes higher as more output ticks are triggered, achieving long-term stability. More precisely, we have:

Theorem 2. *Consider a reset input clock with a given ϵ-tail confidence interval $\left(\mu_\epsilon^{(\text{in})} - \Sigma_\epsilon^{(\text{in})}/2, \mu_\epsilon^{(\text{in})} + \Sigma_\epsilon^{(\text{in})}/2\right)$ satisfying $\Sigma_\epsilon^{(\text{in})} < \mu_\epsilon^{(\text{in})}$. For large enough d, the output ticks of Protocol 2 are i.i.d. with accuracy*

$$R_{\epsilon_0}^{(\text{out})} > \left(R_\epsilon^{(\text{in})} + 1\right) \cdot R_d^{(\text{c})} \tag{12}$$

for any $\epsilon_0 > \epsilon$. Here $R_\epsilon^{(\text{in})} = \mu_\epsilon^{(\text{in})}/\Sigma_\epsilon^{(\text{in})}$ is the input accuracy and $R_d^{(\text{c})}$ is the quantum clock accuracy given by Eq. (8). When the quantum clock is taken to be a Quasi-ideal clock, the accuracy can be further bounded as

$$R_{\epsilon_0}^{(\text{out})} > \frac{1}{2}\left(R_\epsilon^{(\text{in})} + 1\right) \cdot d^{1-\nu} \tag{13}$$

for any $\nu > 0$.

The proof of the above theorem can be found in Appendix B.

V. INCOHERENT PROTOCOLS

In view of the results above, it is natural to ask whether a similar improvement could be achieved by means of classical information processing, namely incoherent protocols that process the input ticks and/or the internal clock's ticks to produce a more accurate output signal. While we do not have a general bound, we present here two natural protocols and show that accuracy enhancement is possible, but with a lower factor. For simplicity, we omit tail probabilities in the following discussion, which can be regarded to be fixed.

First, we show a simple protocol using a d-dimensional classical memory that improves the input signal's accuracy by a factor of $\sqrt{d}$. For any input signal, this protocol produces an output signal according to the following rules:

Protocol 3 An incoherent protocol achieving accuracy enhancement by bunching input clock ticks.

(Initialization) Set the classical memory to the state $c = 0$;
when receiving an input tick:
if $c = d - 1$, **then** produce an output tick and reset the memory to the initial state $c = 0$;
else $c \to c + 1$.
end if

This classical memory can also be regarded as a clock driven by input signals. This protocol improves the accuracy of the input signal by *bunching* the input ticks. When the input ticks are i.i.d., the output tick has an accuracy that is proportional to $R^{(\text{in})} \cdot \sqrt{d}$ according to Eq. (3), which also has a product form like that of the coherent protocol. Further noticing that the accuracy of a d-dimensional Quasi-ideal clock scales like d, we can further express the accuracy of this incoherent protocol as $R^{(\text{in})} \cdot \sqrt{R^{(\text{c})}}$ in an approximate sense.

Compared to the coherent protocol, the gain of the input-bunching protocol is $\sqrt{d}$ instead of d. The tail probability of the input-bunching protocol also increases with d unless the input signal has a sub-Gaussian distribution. Moreover, the coherent protocol achieves a *pure enhancement*, in the sense that only the width of the tick's confidence interval is reduced. On the other hand, the input-bunching protocol increases both the time between ticks and the width of the confidence interval, and thus its output signal may not be of the desired form in some situations.

In the following we propose another incoherent protocol, which is capable of preserving the frequency of the input signal. Imagine that we are given an internal clock system with confidence interval $(\mu^{(\text{c})} - \Sigma^{(\text{c})}/2, \mu^{(\text{c})} + \Sigma^{(\text{c})}/2)$ where $\mu^{(\text{c})}$ and $\Sigma^{(\text{c})}$ are much smaller than those of the input signal. This incoherent protocol works by ignoring all ticks of the internal clock but outputting only the one that follows immediately after every input tick, which can be done with the help of an additional classical bit c (This requirement does not affect our analysis, since we care only about the asymptotic scaling of the accuracies). Explicitly, the protocol works in the following manner:

Protocol 4 An incoherent protocol achieving accuracy enhancement by bunching internal clock ticks.

(Initialization) Set the classical memory to the state $c = 0$;
when receiving an input tick, set the classical memory to the state $c = 1$;
when the internal clock ticks:
if $c = 0$, **then** ignore the clock tick (and do nothing);
else $(c = 1)$ produce an output tick and reset the memory to the state $c = 0$.
end if

The accuracy of this incoherent protocol can be evaluated as follows. If the input tick's confidence interval is narrower than the clock's period, the number of ignored ticks can be made a constant. Then the output's accuracy equals the accuracy of j i.i.d. clock ticks bunched together, where $j \propto \mu^{(\text{in})}/\mu^{(\text{c})}$ is the number of ignored ticks. Notice further that the output mean is close to the

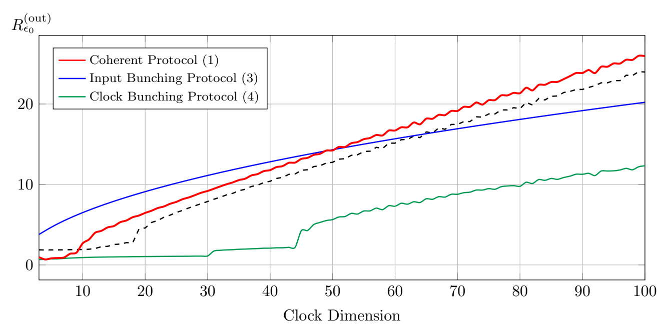

FIG. 3. **Numerical results for comparing the coherent protocol (Protocol 1), the input-bunching protocol (Protocol 3), and the clock-bunching protocol (Protocol 4).** The performances of different accuracy enhancing protocols on an i.i.d. box-shaped input clock of $R_\epsilon^{(\text{in})} \approx 3$ with $\epsilon = 0.01$ are compared. The coherent protocol (red) and the input-bunching protocol (green) use a Quasi-ideal clock of dimension d whilst the input-bunching protocol (blue) uses a d-dimensional classical memory. The allowed error on the output is chosen to be the same as the one on the input (i. e. $\epsilon_0 = 0.01$). The black, dotted line corresponds to the asymptotic lower bound on the output accuracy in Eq. (10), plotted for a fixed clock error $\epsilon^{(\text{c})} = 0.001$. The numerical simulation clearly shows an approximately linear scaling for both the coherent protocol and the clock-bunching protocol and $\sqrt{d}$ scaling for the input-bunching protocol. One can also see that the lower bound in Eq. (10) is quite tight even for low dimensions.

Protocols	Accuracy scaling
Coherent (Protocols 1 and 2)	$R^{(\text{in})} \cdot R^{(\text{c})}$
Bunching input clock ticks (Protocol 3)	$R^{(\text{in})} \cdot \sqrt{R^{(\text{c})}}$
Bunching internal clock ticks (Protocol 4)	$\sqrt{R^{(\text{in})}} \cdot R^{(\text{c})}$

TABLE I. Comparison between the asymptotic performances of coherent and incoherent protocols.

input mean. By Eq. (3), the output tick has accuracy

$$\begin{aligned} R^{(\text{out})} &\approx \sqrt{\frac{\mu^{(\text{in})}}{\mu^{(\text{c})}}} \cdot R^{(\text{c})} \\ &= \sqrt{R^{(\text{in})}} \cdot R^{(\text{c})} \cdot \sqrt{\frac{\Sigma^{(\text{in})}}{\mu^{(\text{c})}}} \\ &\leq \sqrt{R^{(\text{in})}} \cdot R^{(\text{c})}. \end{aligned} \tag{14}$$

From this analysis, we can see that the incoherent protocol is also outperformed by the coherent protocol, since it produces redundant internal clock ticks, and by bunching together these ticks one adds a factor proportional to $\sqrt{\mu^{(\text{in})}/\mu^{(\text{c})}}$ to the width of the confidence interval. In summary, both incoherent protocols lose some accuracy compared to the coherent protocol, which is made apparent in Table I. A numerical simulation, depicted in Fig. 3, shows that the protocols of bunching clock ticks, bunching input ticks and the coherent protocol (introduced in Section III) are all capable of achieving an enhancement, while the performance of the coherent protocol is clearly above that of the other two protocols. Finally, we remark that the same idea works also in the setting where feedback is available.

VI. DISCUSSION AND CONCLUSION

In this work, we proposed protocols that enhance the accuracy of a clock signal by a factor of order d, using a quantum system of dimension d. There are two key properties of the clock that lead to this accuracy enhancement. One is that, for the Quasi-ideal clock, its error is almost independent of its initial position, i.e. when the detector is switched on. The other is that, for most of the time in the protocol, the clock evolves without any dissipation. These two properties originate from the quantumness of the clock system. In this sense, our work justifies the power of quantumness in enhancing the accuracy of clocks. This point is further strengthened via the comparison with two incoherent protocols (cf. Table I).

Our protocols can be used as a subroutine of a highly accurate clock consisting of a macroscopic oscillator producing clock signals and a quantum system that further improves their accuracy. Our protocols may also be employed in a network setting to establish a common clock signal for multiple, and possible distant, nodes, which is crucial for various applications [14–17]. Although each node may be equipped with a clock, these clocks suffer from random drifts as they produce more and more ticks without synchronization. Our accuracy enhancing protocols, especially the one without feedback, fit this task well. To establish synchronized clock signals, the nodes can locally prepare quantum clocks, with the detectors of the clocks controlled by a signal that is broadcast through the network. Then, using our protocol, even if the nodes receive the broadcast signal at different time, their local ticks can still be triggered at almost the same time. In the mean time, since the local clocks tick less frequently, the speed that they drift away from each other can be controlled. In this way, our protocol can be used to maintain synchronization within the network efficiently.

The task considered in this work can be regarded as a signal processing task [18], where an input signal is processed by a special-purpose system. The difference between the particular task of clock signal processing as considered here and general signal processing is that, in the former, no time reference other than the input signal is available. Common operations in signal processing, like time shift, are therefore prohibited, which makes the task harder. Our work is the first and a concrete step towards a theory of time signal processing by harnessing quantum mechanics.

ACKNOWLEDGMENTS

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Appendix A: Proof of Theorem 1.

1. A quantum clock with a switch

An autonomous clock [4, 9–12] is identified by a tuple $\left(\rho^{(\mathrm{c})}, \{\mathcal{M}_\delta^{(\mathrm{c}\to\mathrm{ct})}\}_\delta\right)$, where $\rho^{(\mathrm{c})}$ is the clock state and $\{\mathcal{M}_\delta^{(\mathrm{c}\to\mathrm{ct})} : \mathsf{Lin}(\mathcal{H}_\mathrm{c}) \to \mathsf{Lin}(\mathcal{H}_\mathrm{c}\otimes\mathcal{H}_\mathrm{t})\}_\delta$ is a family of completely positive trace-preserving maps determining the dynamics of the clock and the production of ticks. Here $\mathcal{H}_\mathrm{c}$ is the Hilbert space of the clock state and $\mathcal{H}_\mathrm{t}$ is the tick space. As its name suggests, an autonomous clock produces ticks without any additional input signal as reference. An example of autonomous clocks that will be useful in our work is the Quasi-ideal clock, and more details of the autonomous clock model can be found in [4].

To make use of the input signal, available in our setting, we need to extend the model of autonomous clocks. For this purpose, we introduce the structure of a quantum clock with a switch, which serves as a key ingredient of our protocol. For this kind of clocks, the state space of the clock can be factorized as $\mathcal{H}_\mathrm{c} = \mathcal{H}_\mathrm{b}\otimes\mathcal{H}$. The corresponding clock state is of the form $b\otimes\rho$, where ρ is the clock state of an autonomous clock and $b \in \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ is a control bit. The dynamics map $\mathcal{M}_\delta^{(\mathrm{c}\to\mathrm{ct})}$ is of the control form

$$\mathcal{M}_\delta^{(\mathrm{c}\to\mathrm{ct})} = |0\rangle\langle 0|\otimes\mathcal{U}_\delta + |1\rangle\langle 1|\otimes\mathcal{M}_\delta^{(\mathrm{auto})} \tag{A1}$$

where $\mathcal{U}_\delta$ is infinitesimal form of the clock's unitary dynamics: $\mathcal{U}_\delta(\rho) = (\rho - i\delta[H,\rho])\otimes|0\rangle\langle 0|$ for every operator ρ on $\mathcal{H}$, and $\mathcal{M}_\delta^{(\mathrm{auto})}$ is the dynamics of the autonomous clock. From the above description we can see that the tick production is *switched off* and the clock evolves unitarily if the control bit is zero while the tick production is *switched on* and the clock runs in the same pattern as the autonomous clock if the control bit is one. We denote by $\tau^{(\mathrm{c})}$ the period of the unitary dynamics of the autonomous clock and by ρ_{t^*} the state of the Quasi-ideal clock with $t^* \in (-\tau^{(\mathrm{c})}/2, \tau^{(\mathrm{c})}/2]$. By definition, if the state starts in ρ_{t^*} and evolves unitarily for $\tau^{(\mathrm{c})}$ it will end up in the original state ρ_{t^*}. In an autonomous clock, the clock starts with its state in the initial state Ψ_0. The time it takes to tick is a random variable, denoted here as $T^{(\mathrm{c})}$. When the clock has high enough dimension, $T^{(\mathrm{c})}$ is close to $\tau^{(\mathrm{c})}/2$ with high probability. After producing a tick, the clock will be reset to its original state Ψ_0 [4].

In the setting of our protocol there are input ticks that offer additional time reference to the clock. An input tick will trigger an operation on the control bit. In general, this operation could be any quantum channel $\mathcal{C}^{(\mathrm{b}\to\mathrm{b})}$. In this work, we only consider simple logic operations, which will switch on/off the production of the autonomous clock's ticks. For instance, in our protocols, an input tick aways triggers the detector to be on, which corresponds to be operation $b \to 1$ for $b \in \{0, 1\}$. Notice that the implementation time of these operations is assumed to be zero (which is otherwise arbitrary).

2. Conventions and notions.

For a fixed $\epsilon > 0$, we consider an input signal with i.i.d. input ticks and a confidence interval $\left(\mu_\epsilon^{(\mathrm{in})} - \Sigma_\epsilon^{(\mathrm{in})}/2, \mu_\epsilon^{(\mathrm{in})} + \Sigma_\epsilon^{(\mathrm{in})}/2\right)$ satisfying

$$\Sigma_\epsilon^{(\mathrm{in})} < \frac{2\mu_\epsilon^{(\mathrm{in})}}{3}. \tag{A2}$$

We denote by $R_\epsilon^{(\mathrm{in})} := \mu_\epsilon^{(\mathrm{in})}/\Sigma_\epsilon^{(\mathrm{in})}$ its accuracy.

Before going into the proof, it is convenient to define a couple of variables that we are going to encounter frequently. We denote by $t_i^{(\mathrm{out})}$ the time when the i-th tick of the output signal is produced, and by $t_i^{(\mathrm{in})}$ the time of when the first input tick arrives after the $(i-1)$-th output tick. Notice that, by this definition, $t_i^{(\mathrm{in})} - t_{i-1}^{(\mathrm{in})}$ is no longer i.i.d. since there may be multiple input ticks in between. In particular, we define $t_1^{(\mathrm{in})}$ as the time of arrival for the first input tick. The purpose of this definition is that there may be multiple input ticks that arrive in between two consecutive output ticks but only the first one of them triggers a non-trivial operation on the clock. The time between i-th input tick and $(i+1)$-th input tick is a random variable, denoted as $T_i^{(\mathrm{in})}$. For convenience of analysis, we choose $\tau^{(\mathrm{c})}$ so that

$$\mu_\epsilon^{(\mathrm{in})} = (m + 1/2) \cdot \tau^{(\mathrm{c})} \tag{A3}$$

with $m \in \mathbb{N}^*$. The choice of m will affect the performance of the protocol and will be discussed later. For readers' convenience, notations that appear frequently in the next subsection are listed in Table II.

Notation	Definition
$t_i^{(\mathrm{out})}$	the time when the i-th output tick is produced
$t_1^{(\mathrm{in})}$	the arrival time of the first input tick
$t_i^{(\mathrm{in})}$ $(i \geq 2)$	the arrival time of the first input tick after the $(i-1)$-th output tick
t_i^*	the clock parameter at time $t_i^{(\mathrm{in})}$
$T_i^{(\mathrm{c})}$	$t_i^{(\mathrm{out})} - t_i^{(\mathrm{in})}$
$T^{(\mathrm{in})}$	the time in between two consecutive input ticks
$\mathsf{C}_{i,\epsilon}^{(x)}$ $(x = \mathrm{in}, \mathrm{out})$	the confidence interval of $t_i^{(x)}$
$\Sigma_\epsilon^{(\mathrm{in})}$	width of the confidence interval of input ticks
$\mu_\epsilon^{(\mathrm{in})}$	center of the input confidence interval $\mathsf{C}_\epsilon^{(\mathrm{in})}$
$\tau^{(\mathrm{c})}$	period of unitary evolution of the clock ($\tau^{(\mathrm{c})} \approx 2\mu^{(\mathrm{c})}$)

TABLE II. Table of notations.

3. Confidence intervals of the output ticks

From this subsection, we start to bound the accuracy of the first $j \in \mathbb{N}^*$ ticks, which requires us to determine a confidence interval of the random variable $t_{j+1}^{(\mathrm{out})} - t_1^{(\mathrm{out})}$. The key step of the proof is to show that all the input ticks and the clock ticks, characterized by the random variables $t_i^{(\mathrm{in})}$ and $t_i^{(\mathrm{out})}$ $(i \leq j+1)$, fall within narrow confidence intervals around their expectations with large probability. In this case, the output ticks are generated *as expected.* In the case where either an input tick or an output tick falls outside its respective confidence interval, the output ticks may be triggered too early or too late, resulting in a large error. This idea is made precise by the following lemma:

Lemma 1. *Assume for convenience $t_1^{(\mathrm{in})} = 0$, which is otherwise arbitrary. For $j \in \mathbb{N}^*$ such that*

$$j < \frac{\tau^{(\mathrm{c})} - \Sigma_d^{(\mathrm{c})}}{\Sigma_d^{(\mathrm{c})} + \Sigma_\epsilon^{(\mathrm{in})}}, \tag{A4}$$

the probability that $t_i^{(\mathrm{in})} \in \mathsf{C}_i^{(\mathrm{in})}$ *and* $t_i^{(\mathrm{out})} \in \mathsf{C}_i^{(\mathrm{out})}$ *holds for every* $i \le j+1$ *is at least* $1 - \epsilon_j^{(\mathrm{out})}$*, where the confidence intervals for the input tick and the output tick are defined as*

$$\mathsf{C}_i^{(\mathrm{in})} := (i-1) \cdot \left(\mu_\epsilon^{(\mathrm{in})} - \frac{\Sigma_\epsilon^{(\mathrm{in})}}{2}, \mu_\epsilon^{(\mathrm{in})} + \frac{\Sigma_\epsilon^{(\mathrm{in})}}{2} \right) \tag{A5}$$

and

$$\mathsf{C}_i^{(\mathrm{out})} := \begin{cases} \left(\frac{\tau^{(\mathrm{c})} - \Sigma^{(\mathrm{c})}}{2}, \frac{\tau^{(\mathrm{c})} + \Sigma^{(\mathrm{c})}}{2} \right) & i = 1 \\ (i-1) \cdot \left(\mu_\epsilon^{(\mathrm{in})} - \frac{\Sigma^{(\mathrm{c})}}{2}, \mu_\epsilon^{(\mathrm{in})} + \frac{\Sigma^{(\mathrm{c})}}{2} \right) + t_1^{(\mathrm{out})} & i \ge 2 \end{cases} . \tag{A6}$$

Here $a \cdot (b,c) + d$ *is a shorthand for* $(ab+d, ac+d)$*. The tail probability of the output confidence interval is bounded as*

$$\epsilon_j^{(\mathrm{out})} \le j \cdot \left(\epsilon + \epsilon_d^{(\mathrm{c})} \right) \tag{A7}$$

where ϵ *is the tail probability of the input ticks and* $\epsilon_d^{(\mathrm{c})}$ *is the tail probability of the quantum clock that vanishes as* $d \to \infty$*.*

It is straightforward to check that the conditions (A3) and (A4) guarantee that all the confidence intervals have no intersection and are temporally ordered. If all input and output ticks are in their own confidence intervals, proper causal order among them will be ensured, i.e.

$$t_1^{(\mathrm{in})} \le t_1^{(\mathrm{out})} \le t_2^{(\mathrm{in})} \le \cdots \le t_j^{(\mathrm{out})} \le t_{j+1}^{(\mathrm{in})} \le t_{j+1}^{(\mathrm{out})}. \tag{A8}$$

From Eq. (A6) we can see that the width of an output tick's confidence interval is $\Sigma_d^{(\mathrm{c})}$, which is a quantity dependent on the quantum clock and independent of the input tick. This feature is key to the accuracy enhancement, which is a result of the clock condition.

Proof of Lemma 1. Applying the chain rule, we can express the probability that all ticks are in their confidence intervals as

$$\begin{aligned} \Pr\left[\bigcap_{i=1}^{j+1} \left(t_i^{(\mathrm{in})} \in \mathsf{C}_i^{(\mathrm{in})} \cap t_i^{(\mathrm{out})} \in \mathsf{C}_i^{(\mathrm{out})} \right) \right] = \prod_{i=1}^{j+1} \Bigg(& \Pr\left[t_i^{(\mathrm{in})} \in \mathsf{C}_i^{(\mathrm{in})} \mid \bigcap_{l=1}^{i-1} \left(t_l^{(\mathrm{in})} \in \mathsf{C}_l^{(\mathrm{in})} \cap t_l^{(\mathrm{out})} \in \mathsf{C}_l^{(\mathrm{out})} \right) \right] \\ & \times \Pr\left[t_i^{(\mathrm{out})} \in \mathsf{C}_i^{(\mathrm{out})} \mid \bigcap_{l=1}^{i-1} \left(t_l^{(\mathrm{in})} \in \mathsf{C}_l^{(\mathrm{in})} \cap t_l^{(\mathrm{out})} \in \mathsf{C}_l^{(\mathrm{out})} \right) \cap t_i^{(\mathrm{in})} \in \mathsf{C}_i^{(\mathrm{in})} \right] \Bigg). \end{aligned} \tag{A9}$$

First we bound the probability for $t_i^{(\mathrm{in})}$ to be in its confidence interval given that all previous ticks are within their confidence intervals. Notice that $t_1^{(\mathrm{in})} = 0 \in \mathsf{C}_1^{(\mathrm{in})}$ trivially holds. For $i \ge 2$, we stress that the probability of $t_i^{(\mathrm{in})} \in \mathsf{C}_i^{(\mathrm{in})}$ does not simply follow from the i.i.d. property of the input ticks, because there may be other input ticks in between $t_{i-1}^{(\mathrm{out})}$ and $t_{i-1}^{(\mathrm{in})} \in \mathsf{C}_{i-1}^{(\mathrm{in})}$. Instead, consider the time of arrival of the next input tick after $t_{i-1}^{(\mathrm{in})}$, conditioned on $t_{i-1}^{(\mathrm{out})} \in \mathsf{C}_{i-1}^{(\mathrm{out})}$ and $t_{i-1}^{(\mathrm{in})} \in \mathsf{C}_{i-1}^{(\mathrm{in})}$. Since $\mu^{(\mathrm{in})} - \Sigma_\epsilon^{(\mathrm{in})}/2 > \tau^{(\mathrm{c})} > t_{i-1}^{(\mathrm{out})} - t_{i-1}^{(\mathrm{in})}$ [see Eq. (A3) and Eq. (A4)], if the time it takes this tick to arrive is $T^{(\mathrm{in})} \in (\mu^{(\mathrm{in})} - \Sigma_\epsilon^{(\mathrm{in})}/2, \mu^{(\mathrm{in})} + \Sigma_\epsilon^{(\mathrm{in})}/2)$, then it is clear that this tick will be the first input tick after $t_{i-1}^{(\mathrm{out})}$. In formula, this argument reads

$$\Pr\left[t_i^{(\mathrm{in})} \in \mathsf{C}_i^{(\mathrm{in})} \mid \bigcap_{l=1}^{i-1} \left(t_l^{(\mathrm{in})} \in \mathsf{C}_l^{(\mathrm{in})} \cap t_l^{(\mathrm{out})} \in \mathsf{C}_l^{(\mathrm{out})} \right) \right] \ge 1 - \epsilon. \tag{A10}$$

Now we bound the probability for $t_i^{(\mathrm{out})}$ to be in their confidence intervals, conditioned on all the previous ticks are in their confidence intervals. Define $t_i^* \in (-\tau^{(\mathrm{c})}/2, \tau^{(\mathrm{c})}/2]$ to be so that the clock state is $\Psi_{t_i^*}$ at $t_i^{(\mathrm{in})}$. Intuitively, it is the location of the clock's "hand" at time $t_i^{(\mathrm{in})}$. Since the clock is in the initial state Ψ_0 at time $t_{i-1}^{(\mathrm{out})}$, we obtain the following relation

$$t_i^{(\mathrm{in})} - t_{i-1}^{(\mathrm{out})} = t_i^* + m_i \cdot \tau^{(\mathrm{c})} \tag{A11}$$

where $m_i \in \mathbb{N}$ is the number of periods that the clock has been evolving unitarily. Notice that when $t_i^{(\mathrm{in})} \in \mathsf{C}_i^{(\mathrm{in})}$ and $t_{i-1}^{(\mathrm{out})} \in \mathsf{C}_{i-1}^{(\mathrm{out})}$, we have

$$t_i^* + m_i \cdot \tau^{(\mathrm{c})} \in \left(m \cdot \tau^{(\mathrm{c})} - \frac{(i-1)(\Sigma_d^{(\mathrm{c})} + \Sigma_\epsilon^{(\mathrm{in})})}{2}, m \cdot \tau^{(\mathrm{c})} + \frac{(i-1)(\Sigma_d^{(\mathrm{c})} + \Sigma_\epsilon^{(\mathrm{in})})}{2} \right). \tag{A12}$$

Since $\frac{(i-1)(\Sigma_d^{(\mathrm{c})} + \Sigma_\epsilon^{(\mathrm{in})})}{2} < \frac{\tau^{(\mathrm{c})} - \Sigma_d^{(\mathrm{c})}}{2}$ holds thanks to Eq. (A4), Eq. (A12) implies that

$$m_i = m \tag{A13}$$

and

$$t_i^* \in \left(-\frac{\tau^{(\mathrm{c})} - \Sigma_d^{(\mathrm{c})}}{2}, \frac{\tau^{(\mathrm{c})} - \Sigma_d^{(\mathrm{c})}}{2} \right). \tag{A14}$$

Next, we denote by $T_i^{(\mathrm{c})} := t_i^{(\mathrm{out})} - t_i^{(\mathrm{in})}$ the time that the clock evolves non-unitarily in between two output ticks, whose value can be determined by t_i^* and the clock condition. Bringing together Eq. (A11), Eq. (A13) and the definition of $T_i^{(\mathrm{c})}$, we have

$$t_i^{(\mathrm{out})} = t_{i-1}^{(\mathrm{out})} + m \cdot \tau^{(\mathrm{c})} + \left(T_i^{(\mathrm{c})} + t_i^* \right). \tag{A15}$$

Noticing that the clock condition is guaranteed by Eq. (A14), we can apply the clock condition to $T_i^{(\mathrm{c})} + t_i^*$, which yields that

$$\Pr\left[t_i^{(\mathrm{out})} - t_{i-1}^{(\mathrm{out})} - m \cdot \tau^{(\mathrm{c})} \in \left(\frac{\tau^{(\mathrm{c})} - \Sigma_d^{(\mathrm{c})}}{2}, \frac{\tau^{(\mathrm{c})} + \Sigma_d^{(\mathrm{c})}}{2} \right) \mid \bigcap_{l=1}^{i-1} \left(t_l^{(\mathrm{in})} \in \mathsf{C}_l^{(\mathrm{in})} \cap t_l^{(\mathrm{out})} \in \mathsf{C}_l^{(\mathrm{out})} \right) \cap t_i^{(\mathrm{in})} \in \mathsf{C}_i^{(\mathrm{in})} \right] \geq 1 - \epsilon^{(\mathrm{c})}. \tag{A16}$$

Since $t_i^{(\mathrm{out})} - t_{i-1}^{(\mathrm{out})} - m \cdot \tau^{(\mathrm{c})} \in \left(\frac{\tau^{(\mathrm{c})} - \Sigma_d^{(\mathrm{c})}}{2}, \frac{\tau^{(\mathrm{c})} + \Sigma_d^{(\mathrm{c})}}{2} \right)$ plus $t_{i-1}^{(\mathrm{out})} \in \mathsf{C}_{i-1}^{(\mathrm{out})}$ imply that $t_i^{(\mathrm{out})} \in \mathsf{C}_i^{(\mathrm{out})}$, we conclude that

$$\Pr\left[t_i^{(\mathrm{out})} \in \mathsf{C}_i^{(\mathrm{out})} \mid \bigcap_{l=1}^{i-1} \left(t_l^{(\mathrm{in})} \in \mathsf{C}_l^{(\mathrm{in})} \cap t_l^{(\mathrm{out})} \in \mathsf{C}_l^{(\mathrm{out})} \right) \cap t_i^{(\mathrm{in})} \in \mathsf{C}_i^{(\mathrm{in})} \right] \geq 1 - \epsilon^{(\mathrm{c})}. \tag{A17}$$

Finally, substituting Eq. (A10) and Eq. (A17) into Eq. (A9), we have

$$\Pr\left[\bigcap_{i=1}^{j+1} \left(t_i^{(\mathrm{in})} \in \mathsf{C}_i^{(\mathrm{in})} \cap t_i^{(\mathrm{out})} \in \mathsf{C}_i^{(\mathrm{out})} \right) \right] \geq 1 - \epsilon_j^{(\mathrm{out})} \tag{A18}$$

where $\epsilon_j^{(\mathrm{out})} \leq j \cdot \left(\epsilon + \epsilon_d^{(\mathrm{c})} \right)$ as desired.

■

Remark 1. *Eq. (A4) puts a limit on how small we can set* $\tau^{(\mathrm{c})}$ *to be. When the clock system has large dimension, the term* $\Sigma_d^{(\mathrm{c})}$ *is very small and Eq. (A4) becomes*

$$\tau^{(\mathrm{c})} > j \cdot \Sigma_\epsilon^{(\mathrm{in})}. \tag{A19}$$

Therefore, the protocol cannot run forever with small error, since the above constraint is always going to be violated when j *is large enough. To address this issue, one can choose to reset the output signal every once in a while.*

4. Accuracy of the output signal

In this subsection, we evaluate the accuracy of the output signal using Lemma 1. First, we emphasize that Eq. (A4) in Lemma 1 holds at least for $j = 1$ when the input signal satisfies the condition (A2), as we have $\tau^{(c)} > \Sigma_\epsilon^{(\text{in})}$ by setting $m = 1$ in Eq. (A3). Noticing that $\Sigma_d^{(c)}$ vanishes in the large d limit, we have $\tau^{(c)} > \Sigma_\epsilon^{(\text{in})} + 2\Sigma_d^{(c)}$ for large enough d, and thus Eq. (A4) holds at least for $j = 1$.

Now we show a lower bound of the output accuracy. It is straightforward from Eq. (A6) that there exists a confidence interval of the $(j+1)$-th output tick with center and width

$$\mu_{\epsilon_j}^{(\text{out})} = j \cdot \mu_\epsilon^{(\text{in})} \qquad \Sigma_{\epsilon_j}^{(\text{out})} = j \cdot \Sigma_d^{(c)}, \tag{A20}$$

which has a tail probability $\epsilon_j^{(\text{out})}$ given by Eq. (A7). Therefore, the output accuracy can be evaluated as

$$R_{j,\epsilon_j}^{(\text{out})} = \frac{\mu_\epsilon^{(\text{in})}}{\Sigma_d^{(c)}} = \frac{\Sigma_\epsilon^{(\text{in})}}{\tau^{(c)}/2} \cdot R_\epsilon^{(\text{in})} \cdot R_d^{(c)}. \tag{A21}$$

Notice that for large enough d Eq. (A4) becomes $j\Sigma_\epsilon^{(\text{in})} < \tau^{(c)}$. Choose m in Eq. (A3) as large as possible so that this inequality "barely holds", in the sense that

$$j\Sigma_\epsilon^{(\text{in})} \in \left[\frac{\mu_\epsilon^{(\text{in})}}{m+3/2}, \frac{\mu_\epsilon^{(\text{in})}}{m+1/2}\right). \tag{A22}$$

Then the ratio between $\Sigma_\epsilon^{(\text{in})}$ and $\tau^{(c)}/2$ can be bounded as $\frac{\Sigma_\epsilon^{(\text{in})}}{\tau^{(c)}/2} \geq \frac{2(m+1/2)}{j(m+3/2)} \geq \frac{6}{5j}$. Substituting it into Eq. (A21), we get the bound (10).

When the quantum clock is a Quasi-ideal clock, the clock accuracy is $R^{(c)} = (d^{1-\eta}/2)\left(1 - O(d^{-2\eta})\right)$, which is an immediate consequence of the following lemma (see Appendix C for its proof):

Lemma 2 (Quasi-ideal clock with arbitrary initial position)**.** *A d-dimensional Quasi-ideal clock satisfies the clock condition. Moreover, the width of the confidence interval is*

$$\Sigma^{(c)} := \left(\gamma + \frac{x_{\text{vr}}}{\pi}\right)\tau^{(c)}. \tag{A23}$$

Here $\gamma = d^{-1+\eta} + O(d^{-1})$ *for any* $\eta > 0$ *[4, Eqs. (F23) and (F24)] and* $x_{\text{vr}} = (1/\pi)d^{\frac{3\eta}{4}-1}$ *[4, Eq. (F22)], thus the leading order term in Eq. (A23) is* $\gamma \cdot \tau^{(c)}$*. The tail probability is*

$$\epsilon^{(c)} = 2\delta\tilde{\epsilon}_V + e^{-\delta} + 3\epsilon_{\text{tail}} + 2\epsilon_{\text{trans}}\left(\tau^{(c)}, d\right). \tag{A24}$$

The major term in Eq. (A24) is $2\delta\tilde{\epsilon}_V = o(\gamma)$ *(see [4, Corollary 9 and Eq. (F240)]), whereas* $\delta = \eta/16$ *(see [4, Eq. (F19)]), the other two overhead terms* ϵ_{tail} *and* ϵ_{trans} *also vanish exponentially in d and are given in Section C.*

Substituting the expression of $R^{(c)}$ for the Quasi-ideal clock into Eq. (10), we get Eq. (11).

Appendix B: Proof of Theorem 2

Here we prove Theorem 2 on the accuracy of the output signal for the protocol with feedback. The proof is similar to the proof of Theorem 1 but essentially simpler thanks to the reset mechanism. First, since the output ticks of the quantum clock are i.i.d., we only need to evaluate the accuracy for $t := t_2^{(\text{out})} - t_1^{(\text{out})}$ and the accuracy of the $(j+1)$-th tick can be estimated from Eq. (3).

We choose $\tau^{(c)}$ so that

$$\mu_\epsilon^{(\text{in})} = m \cdot \tau^{(c)} \tag{B1}$$

with $m \in \mathbb{N}^*$. The key step of the proof is again to show that probability that the second input tick and the second output tick, characterized by the random variables $t_2^{(\text{in})}$ and $t_2^{(\text{out})}$, falls within certain intervals around their expectations with large probability.

1. Confidence intervals of the output tick

In this subsection, we show the following lemma:

Lemma 3. *Assume for convenience $t_1^{(\mathrm{out})} = 0$, which is otherwise arbitrary. For a quantum clock satisfying the clock condition in Definition 2 and the constraint*

$$\Sigma_\epsilon^{(\mathrm{in})} < \tau^{(\mathrm{c})} - \Sigma_d^{(\mathrm{c})}, \tag{B2}$$

the probability that $t_2^{(\mathrm{in})} \in \mathsf{C}_2^{(\mathrm{in})}$ and $t_2^{(\mathrm{out})} \in \mathsf{C}_2^{(\mathrm{out})}$ holds is at least $1 - \epsilon^{(\mathrm{out})}$, where the confidence intervals for the input tick and the output tick are defined as

$$\mathsf{C}_2^{(\mathrm{in})} := \left(\mu_\epsilon^{(\mathrm{in})} - \frac{\Sigma_\epsilon^{(\mathrm{in})}}{2}, \mu_\epsilon^{(\mathrm{in})} + \frac{\Sigma_\epsilon^{(\mathrm{in})}}{2}\right) \tag{B3}$$

and

$$\mathsf{C}_2^{(\mathrm{out})} := \left(\mu_\epsilon^{(\mathrm{in})} + \frac{\tau^{(\mathrm{c})} - \Sigma_d^{(\mathrm{c})}}{2}, \mu_\epsilon^{(\mathrm{in})} + \frac{\tau^{(\mathrm{c})} + \Sigma_d^{(\mathrm{c})}}{2}\right). \tag{B4}$$

The tail probability is bounded as $\epsilon^{(\mathrm{out})} \leq \epsilon + \epsilon_d^{(\mathrm{c})}$.

Proof of Lemma 3. By definition, the probability that $t_2^{(\mathrm{in})}$ in its confidence interval is just bounded as

$$\Pr\left[t_2^{(\mathrm{in})} \in \mathsf{C}_2^{(\mathrm{in})}\right] \geq 1 - \epsilon. \tag{B5}$$

The clock state at $t_2^{(\mathrm{in})}$ is $\Psi_{t_2^*}$, where

$$t_2^* = t_2^{(\mathrm{in})} - m_2 \cdot \tau^{(\mathrm{c})} \in \left(-\frac{\tau^{(\mathrm{c})}}{2}, \frac{\tau^{(\mathrm{c})}}{2}\right] \tag{B6}$$

for some $m_2 \in \mathbb{N}$. Under the condition $t_2^{(\mathrm{in})} \in \mathsf{C}_2^{(\mathrm{in})}$ and (B2), we have

$$m_2 = m \tag{B7}$$

and

$$t_2^* \in \left(-\frac{\tau^{(\mathrm{c})} - \Sigma_d^{(\mathrm{c})}}{2}, \frac{\tau^{(\mathrm{c})} - \Sigma_d^{(\mathrm{c})}}{2}\right). \tag{B8}$$

We can then apply Lemma 2, which yields that

$$\left(T_2^{(\mathrm{c})} + t_2^*\right) \in \left(\frac{\tau^{(\mathrm{c})} - \Sigma_d^{(\mathrm{c})}}{2}, \frac{\tau^{(\mathrm{c})} + \Sigma_d^{(\mathrm{c})}}{2}\right) \tag{B9}$$

with probability $1 - \epsilon^{(\mathrm{c})}$. Here $T_2^{(\mathrm{c})} := t_2^{(\mathrm{out})} - t_2^{(\mathrm{in})}$. Combining Eq. (B6), Eq. (B7) with the above equation, we get

$$t_2^{(\mathrm{out})} = m_2 \cdot \tau^{(\mathrm{c})} + T_2^{(\mathrm{c})} + t_2^* \in \mathsf{C}_2^{(\mathrm{out})}, \tag{B10}$$

which means that Eq. (B9) implies $t_2^{(\mathrm{out})} \in \mathsf{C}_2^{(\mathrm{out})}$. Then we conclude that

$$\Pr\left[t_2^{(\mathrm{out})} \in \mathsf{C}_2^{(\mathrm{out})} \,\middle|\, t_2^{(\mathrm{in})} \in \mathsf{C}_2^{(\mathrm{in})}\right] \geq 1 - \epsilon_d^{(\mathrm{c})}. \tag{B11}$$

Finally, by the chain rule, we have

$$\Pr\left[t_2^{(\mathrm{in})} \in \mathsf{C}_2^{(\mathrm{in})} \cap t_2^{(\mathrm{out})} \cap \mathsf{C}_2^{(\mathrm{out})}\right] = \Pr\left[t_2^{(\mathrm{out})} \cap \mathsf{C}_2^{(\mathrm{out})} \,\middle|\, t_2^{(\mathrm{in})} \in \mathsf{C}_2^{(\mathrm{in})}\right] \Pr\left[t_2^{(\mathrm{in})} \in \mathsf{C}_2^{(\mathrm{in})}\right] \tag{B12}$$

$$\geq \left(1 - \epsilon_d^{(\mathrm{c})}\right)(1 - \epsilon) \tag{B13}$$

and $\epsilon^{(\mathrm{out})} \leq \epsilon + \epsilon^{(\mathrm{c})}$ as desired. ■

2. Accuracy of the output signal

In this subsection, we evaluate the accuracy of the output signal using Lemma 3. First, we emphasis that Eq. (B2) in Lemma 3 holds since $\Sigma_\epsilon^{(\mathrm{in})} < \mu_\epsilon^{(\mathrm{in})}$ by assumption and $\Sigma_d^{(\mathrm{c})}$ vanishes as $d \to \infty$ by the clock condition.

Then we show a lower bound of the output accuracy. It is straightforward from Eq. (B4) that there exists a confidence interval of the output tick with center $\mu_\epsilon^{(\mathrm{in})} + \tau^{(\mathrm{c})}/2$ and width $\Sigma_d^{(\mathrm{c})}$ which has a tail probability $\epsilon^{(\mathrm{out})}$. Therefore, the output accuracy can be evaluated as

$$R_{j,\epsilon_j}^{(\mathrm{out})} = \frac{\mu_\epsilon^{(\mathrm{in})} + \tau^{(\mathrm{c})}/2}{\Sigma_d^{(\mathrm{c})}} > \left(\frac{\Sigma_\epsilon^{(\mathrm{in})}}{\tau^{(\mathrm{c})}/2}\right) \cdot \left(R_\epsilon^{(\mathrm{in})} + \frac{1}{2}\right) \cdot R^{(\mathrm{c})}. \tag{B14}$$

Choose m in Eq. (B1) as large as possible so that Eq. (B2) "barely holds", in the sense that

$$\Sigma_\epsilon^{(\mathrm{in})} \in \left[\frac{\mu_\epsilon^{(\mathrm{in})}}{m+1}, \frac{\mu_\epsilon^{(\mathrm{in})}}{m}\right). \tag{B15}$$

Then the ratio between $\Sigma_\epsilon^{(\mathrm{in})}$ and $\tau^{(\mathrm{c})}/2$ can be bounded as $\frac{\Sigma_\epsilon^{(\mathrm{in})}}{\tau^{(\mathrm{c})}/2} \geq \frac{2m}{m+1} \geq 1$. Substituting it into Eq. (B14), we get the bound (12).

When the quantum clock is a Quasi-ideal clock, the clock accuracy is $R^{(\mathrm{c})} = (d^{1-\eta}/2)\left(1 - O(d^{-2\eta})\right)$. Substituting the expression of $R_d^{(\mathrm{c})}$ for the Quasi-ideal clock into Eq. (12), we get Eq. (13).

Appendix C: Proof of Lemma 2

Define $\rho(t) := |\psi_t\rangle\langle\psi_t|$ where $|\psi_t\rangle := e^{-itH - t\delta\bar{V}_d}|\Psi_{t^*}\rangle$ is the unnormalized clock state. Here $|\Psi_{t^*}\rangle$ is the initial state of the non-unitary evolution, H is the Hamiltonian, $\bar{V}_d$ is a real operator corresponds to the interaction potential that generates output tick, and $\delta > 0$ is the interaction strength. Note that the real part of the exponent causes the norm of the state to decrease so that the state is not normalized. The advantage of using this notation is that $\mathrm{Tr}\left[\rho(t)\right]$ equals the probability that the clock evolves for time t without producing any tick (referred to as the delay function in [4]).

Define $I_\pm := \frac{\tau^{(\mathrm{c})}}{2} - t^* \pm \frac{\Sigma_d^{(\mathrm{c})}}{2}$ as the left boundary and the right boundary of the confidence interval. The probability that the tick is generated in the confidence interval $I = [I_-, I_+]$ can be expressed as

$$\Pr\left[T^{(\mathrm{c})} \in I\right] := \mathrm{Tr}\left[\rho(I_-)\right] - \mathrm{Tr}\left[\rho(I_+)\right]. \tag{C1}$$

The statement of Lemma 2, i. e. Eq. (7), can be rephrased as

$$\mathrm{Tr}\left[\rho(I_-)\right] - \mathrm{Tr}\left[\rho(I_+)\right] \geq 1 - \epsilon_d^{(\mathrm{c})}. \tag{C2}$$

Therefore, to show an upper bound of the tail probability, we need to derive a lower bound on $\mathrm{Tr}\left[\rho(I_-)\right]$ and an upper bound on $\mathrm{Tr}\left[\rho(I_+)\right]$. To achieve this, we first introduce the following lemma, which comes immediately from [4, Lemma 21 and Lemma 22]:

Lemma 4 ([4], Lemma 21 and Lemma 22)**.**

$$\Delta_\mathrm{c}(t) - \epsilon_\mathrm{tail} - \epsilon_\mathrm{trans}(t,d) \leq \mathrm{Tr}\left[\rho(t)\right] \leq \Delta_\mathrm{c}(t) + \epsilon_\mathrm{tail} + \epsilon_\mathrm{trans}(t,d). \tag{C3}$$

Here

$$\begin{aligned}\epsilon_\mathrm{trans}(t,d) &= |t|\frac{d}{\tau^{(\mathrm{c})}}\left(O\left(\frac{\sigma^3}{\bar{v}\sigma^2/d+1}\right)^{1/2} + O\left(\frac{d^2}{\sigma^2}\right)\right)e^{-\frac{\pi}{4}\frac{\alpha_0^2}{(d/\sigma^2+\bar{v})^2}\left(\frac{d}{\sigma}\right)^2} + O\left(|t|\frac{d^2}{\sigma^2}+1\right)e^{-\frac{\pi}{4}\frac{d^2}{\sigma^2}} + O\left(e^{-\frac{\pi}{2}\sigma^2}\right)\\ &= O\left(\frac{|t|}{\tau^{(\mathrm{c})}}e^{-\frac{\pi}{4}d^{\frac{\eta}{8}}}\right)\end{aligned} \tag{C4}$$

as defined in [4, Eq. (F38)] where $\sigma = d^{\eta/2}$ (cf. [4, Eq. (F241)]), $\bar{v}\sigma = d^{1-\eta/16}$ (cf. [4, Eq. (F213)]), and α_0 can be set to one (cf. [4, Eq. (F30)]),

$$\epsilon_\mathrm{tail} = O\left(e^{-\frac{\pi}{2}d^\eta}\right) \tag{C5}$$

as defined in [4, Eqs. (F78) and (F81)],

$$\Delta_{\mathrm{c}}(t) := \sum_{k\in I_\gamma(t^*)} e^{-2\delta\int_k^{k+td/\tau^{(\mathrm{c})}} \mathrm{d}y \bar{V}_d(y)} \psi_{\mathrm{nor}}\left(k - \frac{t^* d}{\tau^{(\mathrm{c})}}\right) \tag{C6}$$

where $I_\gamma(t^*) := \left\{\lfloor t^* d/\tau^{(\mathrm{c})} - \gamma d/2\rfloor, \ldots, \lceil t^* d/\tau^{(\mathrm{c})} + 1 + \gamma d/2\rceil\right\}$. *Here* ψ_{nor} *is a normal distribution satisfying* $\sum_{k\in I_\gamma(t^*)} \psi_{\mathrm{nor}}\left(k - \frac{t^* d}{\tau^{(\mathrm{c})}}\right) \geq 1 - \epsilon_{\mathrm{tail}}$.

The next step is to bound the dominant term $\Delta_{\mathrm{c}}(t)$. By [4, Eq. (F13)], x_{vr} is defined so that

$$1 - \tilde{\epsilon}_V = \int_{-x_{\mathrm{vr}}}^{x_{\mathrm{vr}}} \mathrm{d}x\, \bar{V}_0(x + x_0), \tag{C7}$$

where $\bar{V}_0$ is defined via the relation $\bar{V}_d(x) = \frac{2\pi}{d}\bar{V}_0\left(\frac{2\pi}{d}x\right)$. The relation between $\bar{V}_0$ and $\bar{V}_d$ implies that

$$\int_k^{k+\frac{td}{\tau^{(\mathrm{c})}}} \mathrm{d}y\, \bar{V}_d(y) = \int_{\frac{2\pi k}{d} - x_0}^{\frac{2\pi k}{d} - x_0 + \frac{2\pi t}{\tau^{(\mathrm{c})}}} \mathrm{d}x\, \bar{V}_0(x + x_0). \tag{C8}$$

Here we take the location of the potential to be $x_0 = \pi$. Then we can see that:

1. A sufficient condition for $\int_k^{k+td/\tau^{(\mathrm{c})}} \mathrm{d}y \bar{V}_d(y) \leq \tilde{\epsilon}_V$ to hold is that

$$[-x_{\mathrm{vr}}, x_{\mathrm{vr}}] \subset \left[\frac{2\pi k}{d} - \pi, \frac{2\pi k}{d} - \pi + \frac{2\pi t}{\tau^{(\mathrm{c})}}\right]^c \tag{C9}$$

   holds for every $k \in I_\gamma(t^*)$, which is guaranteed when

$$t \leq \frac{\tau^{(\mathrm{c})}}{2} - t^* - \left(\frac{x_{\mathrm{vr}}}{2\pi} + \frac{\gamma}{2}\right)\tau^{(\mathrm{c})} = I_-. \tag{C10}$$

2. A sufficient condition for $\int_k^{k+td/\tau^{(\mathrm{c})}} \mathrm{d}y \bar{V}_d(y) \geq 1 - \tilde{\epsilon}_V$ to hold is that

$$[-x_{\mathrm{vr}}, x_{\mathrm{vr}}] \subset \left[\frac{2\pi k}{d} - \pi, \frac{2\pi k}{d} - \pi + \frac{2\pi t}{\tau^{(\mathrm{c})}}\right] \tag{C11}$$

   holds for every $k \in I_\gamma(t^*)$, which is guaranteed when

$$t \geq \frac{\tau^{(\mathrm{c})}}{2} - t^* + \left(\frac{x_{\mathrm{vr}}}{2\pi} + \frac{\gamma}{2}\right)\tau^{(\mathrm{c})} = I_+. \tag{C12}$$

The above discussion yields the bounds for $\Delta_{\mathrm{c}}(I_-)$ and $\Delta_{\mathrm{c}}(I_+)$. Explicitly, we have:

$$\Delta_{\mathrm{c}}(I_-) \geq \left(\min_{k\in I_\gamma(t^*)} e^{-2\delta\int_k^{k+td/\tau^{(\mathrm{c})}} \mathrm{d}y \bar{V}_d(y)}\right) \cdot (1 - \epsilon_{\mathrm{tail}}) \tag{C13}$$

$$\geq e^{-2\delta\tilde{\epsilon}_V} \cdot (1 - \epsilon_{\mathrm{tail}}), \tag{C14}$$

and

$$\Delta_{\mathrm{c}}(I_+) \leq \left(\max_{k\in I_\gamma(t^*)} e^{-2\delta\int_k^{k+td/\tau^{(\mathrm{c})}} \mathrm{d}y \bar{V}_d(y)}\right) \tag{C15}$$

$$\leq e^{-2\delta(1-\tilde{\epsilon}_V)}. \tag{C16}$$

Therefore, we have

$$\mathrm{Tr}\left[\rho(I_-)\right] - \mathrm{Tr}\left[\rho(I_+)\right] \geq e^{-2\delta\tilde{\epsilon}_V} \cdot (1 - \epsilon_{\mathrm{tail}}) - e^{-2\delta(1-\tilde{\epsilon}_V)} - 2\epsilon_{\mathrm{tail}} - \epsilon_{\mathrm{trans}}\left(I_-, d\right) - \epsilon_{\mathrm{trans}}\left(I_+, d\right) \tag{C17}$$

$$\geq 1 - 2\delta\tilde{\epsilon}_V - e^{-\delta} - 3\epsilon_{\mathrm{tail}} - 2\epsilon_{\mathrm{trans}}\left(\tau^{(\mathrm{c})}, d\right), \tag{C18}$$

having assumed that $\tilde{\epsilon}_V \leq 1/2$ (which always holds since we consider only the asymptotics). One can easily see from the above equation that the tail probability $\epsilon_d^{(\mathrm{c})}$ is bounded as Eq. (A24).