Data-Driven Inference, Reconstruction, and Observational Completeness of Quantum Devices

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The range of a quantum measurement is the set of outcome probability distributions that can be produced by varying the input state. We introduce data-driven inference as a protocol that, given a set of experimental data as a collection of outcome distributions, infers the quantum measurement which is, i) consistent with the data, in the sense that its range contains all the distributions observed, and, ii) maximally noncommittal, in the sense that its range is of minimum volume in the space of outcome distributions. We show that data-driven inference is able to return a measurement up to symmetries of the state space—as it is solely based on observed distributions—and that such limit accuracy is achieved for any data set if and only if the inference adopts a (hyper)-spherical state space (for example, the classical or the quantum bit). When using data-driven inference as a protocol to reconstruct an unknown quantum measurement, we show that a crucial property to consider is that of observational completeness, which is defined, in analogy to the property of informational completeness in quantum tomography, as the property of any set of states that, when fed into any given measurement, produces a set of outcome distributions allowing for the correct reconstruction of the measurement via data-driven inference. We show that observational completeness is strictly stronger than informational completeness, in the sense that not all informationally complete sets are also observationally complete. Moreover, we show that for systems with a (hyper)-spherical state space, the only observationally complete simplex is the regular one, namely, the symmetric informationally complete set.
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In quantum theory, as a consequence of the Born rule, a measurement can always be seen as a linear mapping from the set of states (i.e., density operators) into the set of probability distributions over the measurement outcomes. In fact, some axiomatic approaches *identify* quantum measurements with the set of such mappings: in such a case, the resulting distribution, i.e., the image of the state of the system undergoing the measurement, receives the natural operational interpretation of distribution over the measurement outcomes [1].

When thinking of measurements as linear mappings, the image of the set of *all* states under a given measurement—also known as the measurement's range—turns out to be a very important mathematical object in quantum measurement theory. For example, given two quantum measurements, the range of one includes the range of the other if and only if the former can simulate the latter by means of a suitable statistical transformation [2–4], independently of the state being measured. Quantum measurements, hence, can be compared by comparing the corresponding ranges, thus establishing a deep connection between quantum measurement theory and the theory of majorization and statistical comparison [5, 6], with ramified consequences in both theory and applications.

In this paper we exploit the correspondence between measurements and their ranges to propose a method to extract information about an unknown quantum measurement, based solely on the outcome distributions observed, *without any knowledge about the exact states that gave rise to such distributions*.

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We first define an inference rule, which formulates in an abstract way the rules that we choose to use when reasoning in the presence of incomplete information. For the problem at hand, such rules accept as input a set of outcome distributions and return as output a set of quantum measurements. For this reason, we name our inference rule "data-driven inference (DDI) of quantum measurements." The measurements inferred via DDI are consistent with the input data and are maximally noncommittal, in the sense that their ranges contain the input data and are of minimum volume in the space of outcome distributions. Then we show that it is possible to construct a real experiment so that DDI leads to the correct assignment for the unknown measurement. The goal here is reminiscent of that of conventional quantum measurement tomography namely, the reconstruction of an unknown measurement from the statistics collected in a sequence of experimental trials. However, while measurement tomography requires the use of a known and trusted state preparator to work, DDI reconstruction only requires the analysis of the bare outcome distributions.

Since the theory of data-driven inference and reconstruction is based on the correspondence between measurements and their ranges, three main problems arise and are addressed in this work:

I) The first problem is to seek for a general method to infer a range given a set of outcome distributions. As a possible solution we propose that the measurement range to be inferred, in the face of a set of experimental data, should be the *smallest one containing all the observed data*. Recalling that the range of a measurement is directly related with the ability to simulate other measurements [2], our principle is equivalent to say that the measurements to be inferred should be the weakest possible, compatibly with the data. Our inference rule hence encapsulates a principle of "self-consistent minimality" that we believe constitutes a natural way to reason in the presence of incomplete information. We show that the *only* systems for which DDI always leads to a unique range for any set of data, among all generalized probabilistic theories [7–13], are those with (hyper)-spherical state space, such as the classical and the quantum bit.

II) The second problem consists of understanding to which extent the correspondence between a measurement and its range can be inverted, that is, to what extent a measurement can be characterized if only its range is given. In this respect we show that the correspondence measurement-range is invertible, but only up to the action of a symmetry transformation leaving the state space of the system invariant. This is something to be expected when directly working in the space of outcome distributions, and we consider this to be a feature, rather than a limitation, of DDI.

III) The third problem is to understand how an experimentalist, in complete control of its laboratory, can produce experimental data, which are rich enough to reconstruct, via DDI, the "correct" measurement. That is, we want to understand whether, in order to recover the correct measurement by DDI, an infinite set of states needs to be prepared and sent through the measurement apparatus, or whether a finite set of states, and possibly the same ones for any measurements, suffice. This problem is analogous to the problem in quantum tomography to construct a set of standard apparatus that work whatever it is to be reconstructed. As the problem in quantum tomography is solved by informationally complete apparatus, the analogous problem in DDI reconstruction is solved by what we call *observationally complete* (OC) apparatus. More precisely, OC sets of states are sets whose image contains the same statistical information as the entire range.

Finally, we show that the property of observational completeness is strictly stronger than informational completeness, thus constituting a new "Bureau of Standards" in terms of DDI reconstruction. To this aim we show that, for systems with (hyper)-spherical state space such as the classical and quantum bits, the only observationally complete simplex is the regular simplex, that is, the *symmetric* informationally complete (SIC) one [14–16]. Data-driven inference and reconstruction, hence, naturally lead to the notion of SIC apparatus *by looking only at the set of outcome distributions*, thus providing a completely new viewpoint on the discussion about SIC apparatus and their "natural occurrence" in quantum theory.

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