Quantum Channels for Mixed Unitary Categories

Robin Cockett, Priyaa V. Srinivasan

April 14, 2019

Abstract

In this article we generalize the CP^∞-construction of dagger monoidal categories to mixed unitary categories, as introduced in [4]. Mixed unitary categories provide a setting, which generalizes (compact) dagger monoidal categories and in which one may study quantum processes of arbitrary dimensions.

We show that the existing results [6] for the CP^∞-construction hold in this more general setting. In particular, we generalize the notion of environment structures to mixed unitary categories and show that the CP^∞-construction on mixed unitary categories is axiomatized by generalized environment structures.

1 Introduction

Dagger compact closed categories (†-KCCs) are fundamental to categorical quantum mechanics (CQM) because they provide a diagrammatic framework to study quantum processes between finite dimensional systems. Owing to their 'compact' nature, many of the insights derived from CQM are applicable to quantum information theory and quantum computation. In order to widen the scope of CQM, there have been efforts [6, 7, 1, 9] to extend the structures in CQM to infinite dimensional systems. In our recent work, we generalized the framework of dagger compact closed categories to what we call the mixed unitary categories (MUCs) [4]. Refer Appendix A for a brief review on MUCs. The rest of the introduction gives the definition of MUCs along with two examples which will be later used in this article.

Definition 1.1. *A* **mixed unitary category (MUC)**, $M : \mathbb{U} \longrightarrow \mathbb{C}$*, is a †-isomix category,* $\mathbb{C}$*, equipped with a †-isomix functor* $M : \mathbb{U} \longrightarrow \mathbb{C}$ *from a unitary category* $\mathbb{U}$ *to* $\mathbb{C}$*.*

If $M : \mathbb{U} \longrightarrow \mathbb{C}$ is an MUC where $\mathbb{C}$ is a $*$-autonomous category, then $M : \mathbb{U} \longrightarrow \mathbb{C}$ is said to be a **$*$-MUC**. If moreover every object in $\mathbb{U}$ has unitary duals, $M : \mathbb{U} \longrightarrow \mathbb{C}$ is said to be a **$*$-mixed unitary category with duals** ($*$-MUdC).

In this article, we use the following two examples of MUCs: $\mathbb{R} \subset \mathbb{C}$ and $\mathsf{Mat}_\mathbb{C} \subset \mathsf{FMat}(\mathbb{C})$.

Consider the discrete monoidal category $\mathbb{C}$ of complex numbers which is defined as follows:

Objects: $a + ib \in \mathbb{C}$

Maps: Identity maps only $c = c$

Tensor: multiplication $(a + ib) \otimes (x + iy) := (ax - by) + i(ay + bx)$

Unit: 1

Dagger: $(a + ib)^\dagger := a - ib$

$\mathbb{C}$ is a compact LDC ($\otimes \simeq \oplus$) with a non-stationary dagger functor.

The subcategory $\mathbb{R}$ of real numbers is a unitary category with the unitary structure map being the identity map. Thus, $\mathbb{R} \subset \mathbb{C}$ is a mixed unitary category.

The second example is using finiteness spaces. A **finiteness space**, $(X, \mathcal{A}, \mathcal{B})$, consists of a set X and a subset $\mathcal{A}, \mathcal{B} \subseteq P(X)$ such that $\mathcal{B} = \mathcal{A}^{\perp}$, that is

$$\mathcal{B} = \{b | b \in P(X) \text{ with for all } a \in A, |a \cap b| < \infty\},$$

and $\mathcal{A} = \mathcal{B}^{\perp}$.

A **finiteness relation**, $(X, \mathcal{A}, \mathcal{B}) \xrightarrow{R} (Y, \mathcal{A}', \mathcal{B}')$ is relation $X \xrightarrow{R} Y$ such that

$$\forall A \in \mathcal{A}. AR \in \mathcal{A}' \quad \text{and} \quad \forall B' \in \mathcal{B}'. RB' \in \mathcal{B}$$

Finiteness spaces with finiteness relation form a $*$-autonomous category.

Objects: Finiteness spaces $(X, \mathcal{A}, \mathcal{B})$

Maps: $(X, \mathcal{A}, \mathcal{B}) \xrightarrow{M} (Y, \mathcal{A}', \mathcal{B}')$ is a matrix $XxY \xrightarrow{M} \mathbb{C}$ such that

$$supp(M) := \{(x, y) | x \in X, y \in Y \text{ and } M(x, y) \neq 0\}$$

is a finiteness relation from $(X, \mathcal{A}, \mathcal{B})$ to $(Y, \mathcal{A}', \mathcal{B}')$.

Dagger: $(X, \mathcal{A}, \mathcal{B})^{\dagger} := (X, \mathcal{B}, \mathcal{A})$. $M^{\dagger}$ is the complex conjugate of M.

$\mathsf{Mat}_{\mathbb{C}}$, the category of finite matrices over complex numbers is a full subcategory of $\mathsf{FMat}(\mathbb{C})$ which is determined by the objects, $(X, P(X), P(X))$, where X is a finite set. $\mathsf{Mat}_{\mathbb{C}}$ is a unitary category, indeed a well-known $\dagger$-compact closed category. The inclusion $\mathsf{Mat}_{\mathbb{C}} \subset \mathsf{FMat}(\mathbb{C})$ is a mixed unitary category.

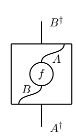
As the next step, in this paper, we aim to generalize the CP^{∞} construction on dagger monoidal categories to the mixed unitary categories. We first introduce a circuit calculus for mixed unitary categories which is an extension of proof nets of linearly distributive categories and is similar to the pictures used in classical categorical quantum mechanics. The extended calculus makes the proofs tractable and more readable.

2 String Calculus for Mixed Unitary Categories

In order to facilitate reasoning within MUCs, it is useful to employ a circuit calculus. We build on the circuit calculus for LDCs introduced in [3]. The extended circuit calculus for mixed unitary categories includes dagger boxes, components for unitary structure maps and inverse mixor morphisms.

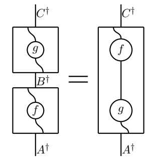
2.1 Dagger functor box

Suppose $\mathbb{X}$ is a $\dagger$-LDC and $f : A \to B \in \mathbb{X}$. Then, the map $f^{\dagger} : B^{\dagger} \to A^{\dagger}$ is graphically depicted as follows:

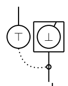


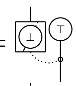
The rectangle is a functor box for the $\dagger$-functor. Notice how we use vertical mirroring to express the contravariance of the $\dagger$-functor. By the functoriality of $(\_)^{\dagger}$, we have: ф $=$ |.

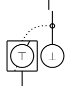
These contravariant functor boxes compose... contravariantly. Given maps $f : A \longrightarrow B$ and $g : B \longrightarrow C$:

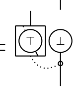


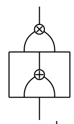
The following are the representations of the basic natural isomorphisms of a †-LDC:

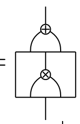
$\lambda_\top : \top \to \bot^\dagger =$ 

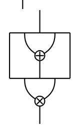
$\lambda_\top^{-1} : \bot^\dagger \to \top =$ 

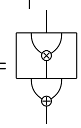
$\lambda_\bot : \bot \to \top^\dagger =$ 

$\lambda_\bot^{-1} : \top^\dagger \to \bot =$ 

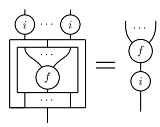
$\lambda_\otimes : A^\dagger \otimes B^\dagger \to (A \oplus B)^\dagger =$ 

$\lambda_\oplus : A^\dagger \otimes B^\dagger \to (A \oplus B)^\dagger =$ 

$\lambda_\otimes^{-1} : (A \oplus B)^\dagger \longrightarrow A^\dagger \otimes B^\dagger =$ 

$\lambda_\oplus^{-1} : (A \otimes B)^\dagger \longrightarrow A^\dagger \oplus B^\dagger =$ 

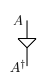
Dagger boxes interact with involutor $A \xrightarrow{\iota} A^{\dagger\dagger}$ as follows:



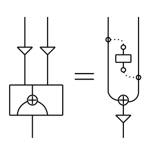
It is important to note that one may not have a legal proof net inside a †-box. This complicates the correctness criterion. However, the required correctness criterion is discussed in [8].

2.2 Unitary structure map

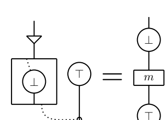
Suppose A is a unitary object, Then, $A \xrightarrow{\varphi_A} A^\dagger$ is drawn as as a downward pointing triangle:



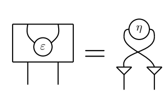
Diagrammatic representations of **[U.3(a)]**, **[U.2(a)]** and **[Udual(a)]** are as follows:



$(\varphi_A \otimes \varphi_B)\lambda_\otimes = \mathsf{mx}\ \varphi_{A \oplus B}$



$\varphi_\bot(\lambda_\top^{-1}) = \mathsf{m}$

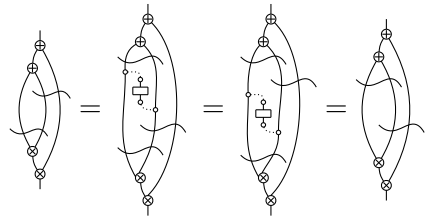


$\lambda_\top \varepsilon_A^\dagger \lambda_\oplus^{-1} = \eta_A \varphi_\oplus \varphi c_\oplus$

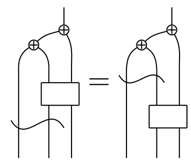
2.3 Inverse of the mixor

Suppose U is an object in the core of $\mathbb{X}$. Recall that by definition, there is a map $U \otimes A \xrightarrow{\mathsf{mx}_{U,A}} U \oplus B$. We graphically represent the inverse of this map as follows:

Observe that mx^{-1} maps asscocitated to different core objects slide past each other:



And, indeed by naturality, mx^{-1} slides over components in circuits:



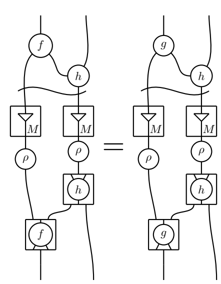
3 Channels for mixed unitary categories

The CPM construction [10] on dagger compact closed categories applied to the concrete category of finite-dimensional Hilbert Spaces and linear maps produces a category of mixed states and quantum processes. Coecke and Heunen [6] generalized the CPM construction to †-symmetric monoidal categories, and thus, to infinite dimensions. They call the generalized construction the CP^∞ construction. In this article, we adapt the CP^∞ construction to our setting of MUCs. We show that our construction coincides with the original CP^∞ construction when the MUC is a †-monoidal categories.

3.1 Kraus maps

A Kraus map $(f, U) : A \to B$ in a mixed unitary category, $M : \mathbb{U} \to \mathbb{C}$, is a map $f : A \to M(U) \oplus B \in \mathbb{C}$ for some $U \in \mathbb{U}$. $M(U)$ is called the ancillary system of f. We glue the Kraus map to its dagger along its ancillary system giving rise to a combinator acting on so called "test maps" to establish an equivalence relation on Kraus maps:

Definition 3.1. *Given a MUC, $\mathbb{U} \xrightarrow{M} \mathbb{C}$, two Kraus maps $(f, U), (g, V) : A \to B$ are* **equivalent**, *$(f, U) \sim (g, V)$, if for all unitary objects X and all maps $h : B \otimes C \to V$ (called* **test maps***), the following equation holds:*



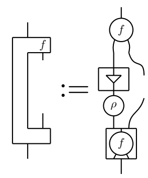
Note that the mx^{-1} map can be slided up and down along the wires of the unitary objects $M(U)$ and $M(V)$ by naturality of the mx map. The diagram includes covariant functor boxes for M and contravariant functor boxes for the dagger. The left hand side diagram is given equationally as follows, making both of the previous diagrams proof nets:

$$A \otimes C \xrightarrow{f \otimes 1} (X \oplus B) \otimes C \xrightarrow{\delta} X \oplus (B \otimes C) \xrightarrow{1 \oplus h} M(U) \oplus M(V) \xrightarrow{\mathsf{mx}^{-1}} M(U) \otimes M(V)$$

$$\xrightarrow{M(\varphi_U)\otimes M(\varphi_V)} M(U^\dagger)\otimes M(V^\dagger) \xrightarrow{\rho\otimes\rho} M(U)^\dagger\otimes M(V)^\dagger \xrightarrow{1\otimes(h^\dagger\lambda_\oplus^{-1})} M(U)^\dagger\otimes(B^\dagger\oplus C^\dagger)$$

$$\xrightarrow{\delta} (M(U)^\dagger\otimes B^\dagger)\oplus C^\dagger \xrightarrow{\lambda_\otimes\oplus 1} (M(U)\oplus B)^\dagger\oplus C^\dagger \xrightarrow{f^\dagger\oplus 1} A^\dagger\oplus C^\dagger \xrightarrow{\lambda_\oplus} (A\otimes C)^\dagger$$

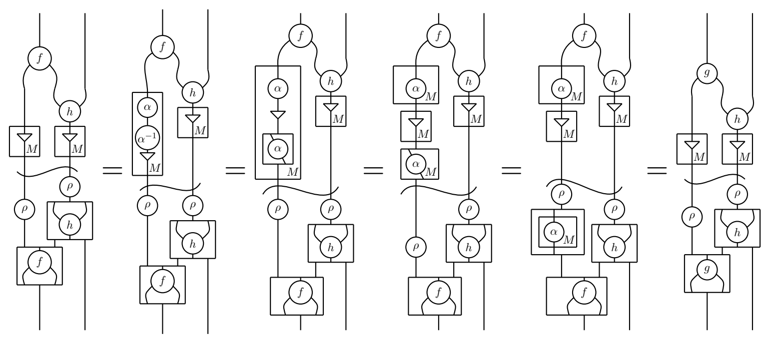
By forgetting the test maps and glueing Kraus map with its dagger, one gets a notationally convinient combinator:



An equivalence class of Kraus morphisms is a **quantum channel**. If $(f, U) : A \longrightarrow B$ and $(g, V) : A \longrightarrow B$ are Kraus maps for which U and V are unitarily isomorphic, they are necessarily equivalent with respect to this relation:

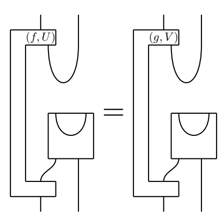
Lemma 3.2. *Let $(f, U), (g, V) : A \longrightarrow B$ be Krauss morphisms. If $U \xrightarrow{\alpha} V$ is a unitary isomorphism and $f(\alpha \oplus 1) = g$, then $f \simeq g$.*

Proof. Let $h : B \otimes C \longrightarrow M(X)$ be any test map. Then,

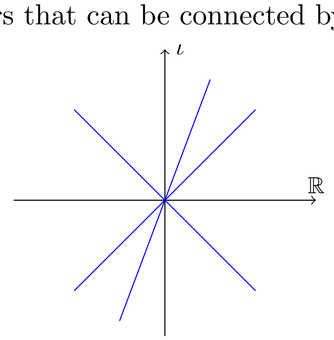


□

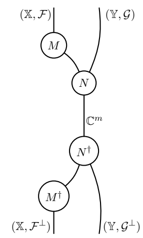
Lemma 3.3. *Suppose $M : \mathbb{U} \longrightarrow \mathbb{C}$, is a $*$-MUC, that is every object in $\mathbb{C}$ has linear adjoint, then any two Krauss maps f and g are equivalent if and only if*



Let us now examine Kraus maps in our running examples for MUCs. Consider the MUC $\mathbb{R} \subset \mathbb{C}$. Let c, c' be any two complex numbers. Kraus maps in $\mathbb{R} \subset \mathbb{C}$ are $(=, r)]c \longrightarrow c'$ such that $c = rc'$. If $c' \neq 0$, then there is at most one Kraus map $(=, r) : c \longrightarrow c'$, for all $c \in \mathbb{C}$ and $c = c'$. If $c' = 0$, then $c = 0$ and for all $r' \in \mathbb{R}$, $(=, r) \sim (=, r') : c \longrightarrow c'$. It can be observed that there are Kraus maps only between those complex numbers that can be connected by a line drawn from the origin:



A Kraus map $(M, \mathbb{C}^m) : (X, \mathcal{A}, \mathcal{A}^{\perp}) \to (Y, \mathcal{B}, \mathcal{B}^{\perp})$ gives a **pure completely positive map**:



Choi's theorem states that every completely positive map can be written as a sum of pure completely positive maps. As a consequence of the theorem, every map in the category resulting from CP^{∞} construction on the MUC $\mathsf{FMat} \subset \mathsf{Mat}$ can be written as a sum of pure maps. The CP^{∞} construction is as follows:

Definition 3.4. *Given a mixed unitary category, $M : \mathbb{U} \to \mathbb{C}$, define $\mathsf{CP}^{\infty}(M : \mathbb{U} \to \mathbb{C})$ to have:*

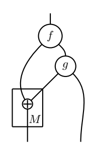
Objects: *Same as $\mathbb{C}$*

Maps: $[(f, U)] : A \to B \in \mathsf{CP}^{\infty}(M : \mathbb{U} \to \mathbb{C})$ *is Kraus maps $(f, U) : A \to B \in \mathbb{X} \;/ \sim$*

Composition: *Given maps $[(f, U)] : A \to B$ and $[(g, V)] : B \to C$ in $\mathsf{CP}^{\infty}(\ {}_{\mathbb{U}}\mathbb{C}_M)$, composition is defined as*

$$[(f, U)][(g, V)] := A \xrightarrow{f} U \oplus B \xrightarrow{1 \oplus g} U \oplus (V \oplus C) \xrightarrow{a_{\oplus}} (U \oplus V) \oplus C \in \mathbb{C}$$

Graphically, this is represented by the following map in $\mathbb{C}$:



Identity: 1_A *is defined as the Kraus map* $A \xrightarrow{[(u_{\oplus}^{L})^{-1}]} \perp \oplus A \xrightarrow{\simeq} M(\perp) \oplus A \in \mathbb{X}$

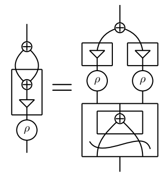
To prove that $\mathsf{CP}^{\infty}(M : \mathbb{U} \to \mathbb{C})$ is a category, we first observe the following result about the unitary objects. Applying unitary structure maps on two objects seperately is equivalent to applying one unitary structure map on the compound object:

Lemma 3.5. *The following diagarms commute:*

$$\begin{array}{ccc}
M(C) \otimes M(D) & \xrightarrow{M(\varphi) \otimes M(\varphi)} & M(C^{\dagger}) \otimes M(D^{\dagger}) \\
\downarrow{\scriptstyle m_{\otimes}^{M}} & & \downarrow{\scriptstyle \rho \otimes \rho} \\
M(C \otimes D) & & M(C)^{\dagger} \otimes M(D)^{\dagger} \\
\downarrow{\scriptstyle M(\varphi)} & \textbf{(a)} & \downarrow{\scriptstyle \lambda_{\otimes}} \\
M((C \otimes D)^{\dagger}) & & (M(C) \oplus M(D))^{\dagger} \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \mathsf{mx}^{\dagger}} \\
(M(C) \otimes M(D))^{\dagger} & \xrightarrow[(m_{\otimes}^{M})^{\dagger}]{} & (M(C) \otimes M(D))^{\dagger}
\end{array}
\qquad
\begin{array}{ccc}
M(C) \oplus M(D) & \xrightarrow{M(\varphi_C) \oplus M(\varphi_D)} & M(C^{\dagger}) \oplus M(D^{\dagger}) \\
\downarrow{\scriptstyle n_{\oplus}^{-1}} & & \downarrow{\scriptstyle \rho \oplus \rho} \\
M(C \oplus D) & & M(C)^{\dagger} \oplus M(D)^{\dagger} \\
\downarrow{\scriptstyle M(\varphi)} & \textbf{(b)} & \downarrow{\scriptstyle \lambda_{\oplus}} \\
M((C \oplus D)^{\dagger}) & & (M(C) \otimes M(D))^{\dagger} \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle (\mathsf{mx}^{-1})^{\dagger}} \\
(M(C \oplus D))^{\dagger} & \xleftarrow[n_{\oplus}^{\dagger}]{} & (M(C) \oplus M(D))^{\dagger}
\end{array}$$

Proof. For proof, refer Appendix B. □

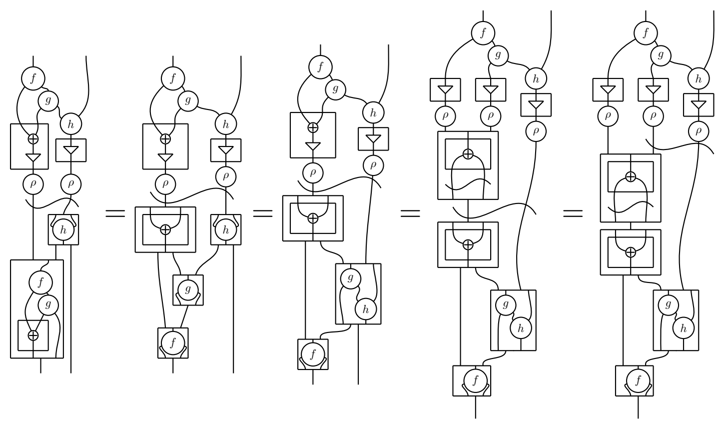
Diagrammatic representation of Lemma B.2 (b):

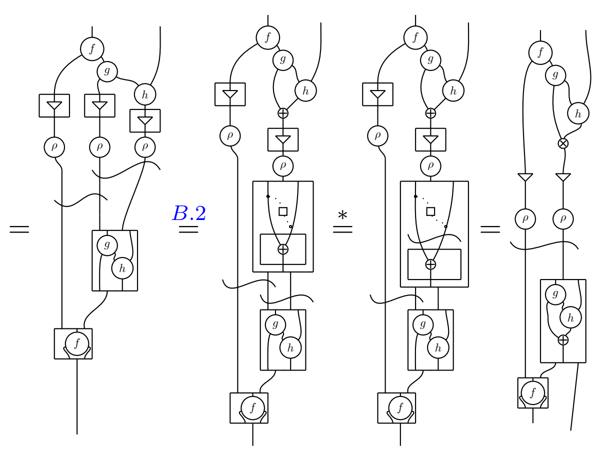


Proposition 3.6. *$\mathsf{CP}^\infty(M : \mathbb{U} \longrightarrow \mathbb{C})$ is a category.*

Proof.

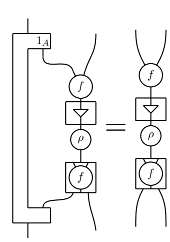
- Composition is well-defined: That is to say, if $(f, U) \sim (f', U')$ then $(f, U)(g, V) \sim (f', U')(g, V)$. First we observe that: It suffices to show that:



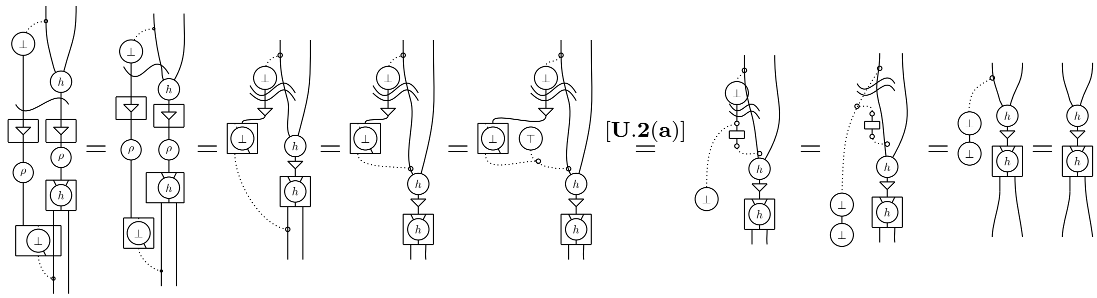


  $(g, V) \sim (g', V') \Rightarrow (f, U)(g, V) \sim (f, U)(g', V')$ is proved similarly.

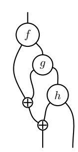
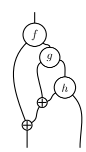
- Identity laws hold: $[(u^L_\oplus)^{-1}][f] = [f]$ It suffices to prove that $(u^L_\oplus)^{-1} f \sim f \in \mathbb{C}$:



The proof is as follows:



- Composition is associative: Suppose $A \xrightarrow{f} B, B \xrightarrow{g} C, C \xrightarrow{h} D \in \mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$. Let U_1, U_2, and U_3 be ancillary systems of f, g and $h \in \mathbb{C}$ respectively. Observe:

$(fg)h :=$  $\sim$  $=: f(gh)$

Since $(U_1 \oplus U_2) \oplus U_3 \xrightarrow{a_\oplus} U_1 \oplus (U_2 \oplus U_3)$ is a unitary isomorphism, by Lemma 3.2, $(fg)h \sim f(gh) \in \mathbb{C} \Rightarrow (fg)h = f(gh) \in \mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$.

□

There is a canonical functor $Q : \mathbb{C} \longrightarrow \mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$ of the original category into the category of channels:

Lemma 3.7. *Let $M : \mathbb{U} \to \mathbb{C}$ be a mixed unitary category, then there is a canonical functor $Q : \mathbb{C} \longrightarrow \mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$:*

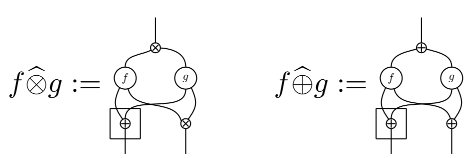
$$\text{On objects:} A \mapsto A$$
$$\text{On maps:} f \mapsto f(u_\oplus^L)^{-1}$$

Proof. Q is a preserves identity maps and composition because $f(u_\oplus^L)^{-1} \sim f \sim (u_\oplus^L)^{-1} f$. □

There is no reason why this functor should be faithful and, indeed, in many cases it will *not* be faithful [6].

Theorem 3.8. $\mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$ *is an isomix category.*

Proof. We know that $\mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$ is well-defined category. Indeed it has two tensors: $\widehat{\otimes}$ and $\widehat{\oplus}$ given by the following Kraus maps:



The units for $\widehat{\otimes}$ and $\widehat{\oplus}$ are $\top$ and $\bot$ respectively.

The linear distribution maps and all the basic basic natural isomorphisms are inherited from $\mathbb{X}$ by composing each one of them with $(u_\oplus^L)^{-1}$ i.e.,

$$\frac{A \otimes (B \otimes C) \xrightarrow{a_\otimes} (A \otimes B) \otimes C \xrightarrow{(u_\oplus^L)^{-1}} \mathbb{C} \ / \sim}{A \otimes (B \otimes C) \xrightarrow{a_{\widehat{\otimes}} := a_\otimes (u_\oplus^L)^{-1}} (A \otimes B) \otimes C \in \mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})}$$

We prove that the associators and the other maps as defined above are natural isomorphisms in $\mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$: From Lemma 3.7, $Q : \mathbb{C} \hookrightarrow \mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$ is functorial which means that all commuting diagrams and isomorphisms are preserved. It remains to show that Q preserves the linear structure and the mix map:

- Q preserves $\otimes$: Suppose $A \xrightarrow{f(u_\oplus^L)^{-1}} A'$ and $B \xrightarrow{g(u_\oplus^L)^{-1}} B'$. Then, $Q(f) \widehat{\otimes} Q(g) = Q(f \otimes g)$:

$$Q(f) \widehat{\otimes} Q(g) := A \otimes B \xrightarrow{f(u_\oplus^L)^{-1} \widehat{\otimes} g(u_\oplus^L)^{-1}} (\bot \oplus \bot) \oplus (A' \otimes B')$$
$$Q(f \otimes g) := A \otimes B \xrightarrow{(f \otimes g) u_\oplus^{-1}} \bot \oplus (A' \otimes B')$$

  Since, $\bot \oplus \bot \xrightarrow{u_\oplus^L} \bot$ is a unitary isomorphism and, $f(u_\oplus^L)^{-1} \widehat{\otimes} g(u_\oplus^L)^{-1}(u_\oplus^L \oplus 1) = (f \otimes g)(u_\oplus^L)^{-1} \in \mathbb{C}$, by Lemma 3.2,

$$f(u_\oplus^L)^{-1}) \widehat{\otimes} (g(u_\oplus^L)^{-1} \sim (f \otimes g)(u_\oplus^L)^{-1}$$

  Therefore, $Q(f) \widehat{\otimes} Q(g) = Q(f \otimes g)$. Similarly, $Q(f) \widehat{\oplus} Q(g) = Q(f \oplus g)$.

- Q preserves all basic natural isomorphisms (associators, unitors, symmetry maps, mix map) and linear distributions:

  To prove that $a_{\widehat{\otimes}}$ is natural in $\mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$, we need to prove that the following diagram commutes in $\mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$:

$$\begin{array}{ccc} A \widehat{\otimes} (B \widehat{\otimes} C) & \xrightarrow{a_{\widehat{\otimes}}} & (A \widehat{\otimes} B) \widehat{\otimes} C \\ {\scriptstyle (f \widehat{\otimes} g) \widehat{\otimes} h} \downarrow & & \downarrow {\scriptstyle f \widehat{\otimes} (g \widehat{\otimes} h)} \\ A' \widehat{\otimes} (B' \widehat{\otimes} C') & \xrightarrow{a_{\widehat{\otimes}}} & (A' \widehat{\otimes} B') \widehat{\otimes} C' \end{array}$$

  In other words, we need to show that the two compositions in $\mathbb{C}$ are equivalent as Kraus maps. This is follows from Lemma 3.2 as there is a unitary isomorphism between the ancillary objects $\bot \oplus (U_1 \oplus (U_2 \oplus U_3))$ and $\bot \oplus (U_1 \oplus U_2) \oplus U_3$. Similarly, we can show that the other basic linearly distrbutive transformations as defined are natural transformations. Since Q is functorial, it preserves isomorphisms and commuting diagrams so that the coherence diagrams automatically commute.

□

Observe that our CP^∞ construction on $M : \mathbb{U} \to \mathbb{C}$ coincides with the original CP^∞ construction [6] when $M : \mathbb{U} \to \mathbb{C}$ is dagger monoidal category .i.e., $M : \mathbb{U} \to \mathbb{C}$ is identity functor.

$\mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$ does not have a canonical dagger even though $\mathbb{C}$ is a $\dagger$-isomix category. However, if $M : \mathbb{U} \to \mathbb{C}$ is a $*$-MUdC i.e., mixed unitary category in which every object in $\mathbb{U}$ has unitary duals and $\mathbb{C}$ is a $\dagger$-isomix $*$-autonomous category, then $\mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$ has an obvious dagger as shown in the following theorem:

Lemma 3.9. *If $M : \mathbb{U} \to \mathbb{C}$ is a $*$-MUdC then $\mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$ is also a $*$-MUdC.*

Proof. For sketch of proof, refer Appendix B. □

The following table summarizes the structures inherited by $\mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$ from $M : \mathbb{U} \to \mathbb{C}$:

$M : \mathbb{U} \to \mathbb{C}$	$\mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$
mixed unitary category	isomix category
$*$-mixed unitary category with unitary duals	$*$-mixed unitary category with unitary duals
†-symmetric monoidal category	symmetric monoidal category
†-compact closed category	†-compact closed category ($\mathsf{CP}^\infty(\mathbb{X}) \simeq CPM(\mathbb{X})$)

4 Environment structure for mixed unitary categories

In this section, we desribe when a given isomix category is of the form $\mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$. We generalize the environment structures [6] for †-symmetric monoidal categories to mixed unitary categories. We follow the foootsteps of [6] in axiomatizing the CP^∞ construction using environment structures.

4.1 Environment structure and examples

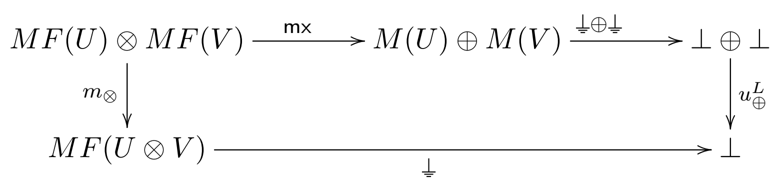
Definition 4.1. *Let $M : \mathbb{U} \to \mathbb{C}$ be a mixed unitary category. An* **environment structure** *for $\mathbb{C}$ is a pair*

$$(F : \mathbb{C} \to \mathbb{D}, ⏚)$$

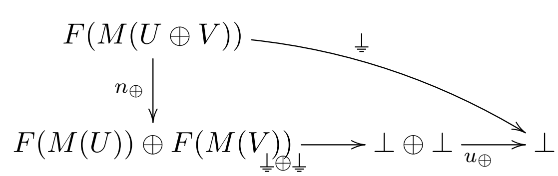
where $\mathbb{D}$ is any isomix category, F is a strict Frobenius isomix functor, and $⏚ : F(M(U)) \to \bot$ is a family of maps indexed by the objects $U \in \mathbb{U}$ such that the following conditions hold:

[Env.1] *For all unitary objects $U, V \in \mathbb{U}$, the following diagrams commute:*

(a)



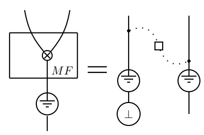
(b)

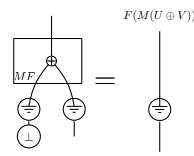


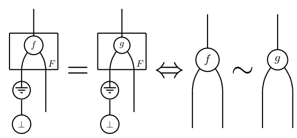
[Env.2] *$f \sim g$ if and only if the following diagram commutes:*

$$
\begin{array}{ccc}
 & F(M(A)) & \\
{}^{F(M(f))}\swarrow & & \searrow^{F(M(g))} \\
F(M(U \oplus B)) & & F(M(U \oplus B)) \\
\downarrow \nu_\otimes & & \downarrow \nu_\otimes \\
F(M(U)) \oplus MF(B) & & F(M(U)) \oplus F(M(B)) \\
\downarrow \, ⏚ \oplus 1 & & \downarrow \, ⏚ \oplus 1 \\
\bot \oplus F(M(B)) & = \!= \!= \!= & \bot \oplus F(M(B))
\end{array}
$$

The conditions are represented diagrammatically as follows:

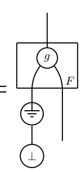
[Env.1a] 

[Env.1b] 

[Env.2] 

Definition 4.2. *Let $M : \mathbb{U} \longrightarrow \mathbb{C}$ be a mixed unitary category and $(F : \mathbb{C} \longrightarrow \mathbb{D}, ⏚)$ be an environment structure for $\mathbb{X}$. $(F : \mathbb{C} \longrightarrow \mathbb{D}, ⏚)$ has* **purification** *if*

- *F is injective on objects, and*
- *for all $f : A \longrightarrow B \in \mathbb{D}$, there exists a Kraus map $f' : X \xrightarrow[U]{} Y \in \mathbb{C}$ such that*

[Env.3] $f =$ 

Equationally,

$$A \xrightarrow{f} B = F(A) \xrightarrow{F(f')} F(M(U) \oplus Y) \xrightarrow{n_\oplus} M(F(U)) \oplus F(Y) \xrightarrow{⏚ \oplus 1} \bot \oplus F(Y) \xrightarrow{u_\oplus} F(Y)$$

Note that $(F : \mathbb{C} \longrightarrow \mathbb{D}, ⏚)$ has purification, F is also surjective on objects since any object $Y \in \mathbb{Y}$ is of the form $F(X)$ for some $X \in \mathbb{X}$. Thus, F is bijective on objects.

Lemma 4.3. *For any mixed unitary category $M : \mathbb{U} \longrightarrow \mathbb{C}$, there exists an environment structure $(Q : \mathbb{C} \longrightarrow \mathsf{CP}^\infty(M : \mathbb{U} \longrightarrow \mathbb{C}), ⏚)$ for $M : \mathbb{U} \longrightarrow \mathbb{C}$ satisfying purification axiom.*

Proof. Q **is a strict isomix functor:** Define $Q : \ C \longrightarrow \mathsf{CP}^\infty(M : \mathbb{U} \longrightarrow \mathbb{C})$ as follows:

$$Q(A) := A$$
$$Q(f) := [(f(u_\oplus^L)^{-1}(n_\bot^M)^{-1}, \bot)]$$

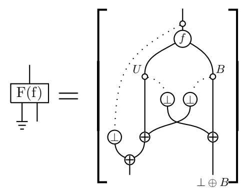
Q is functorial since $f(u_\oplus^L)^{-1} \sim f \sim (u_\oplus^L)^{-1} f$. Define $Q_\otimes = Q_\oplus := Q$. Note that, Q is a strict monoidal functor and an isomix functor.

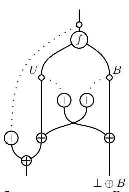
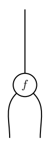
The environment map: For each unitary object $U \in \mathbb{U}$, define

$$⏚ : U \longrightarrow \bot \in \mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C}) := [((u_\oplus^R)^{-1}, U)]$$

In order to prove that the conditions for environment structures are satisfied, properties of isomix functor and Lemma 3.2 are used.

Purification: To prove that $(Q : M : \mathbb{U} \to \mathbb{C} \to \mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C}), ⏚)$ has purification, consider any map $[f] : A \to B \in \mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$. Then, there exists a Kraus map $f : A \xrightarrow[U]{} B \in \mathbb{X}$. Then, the map in equation **[Env. 3]** has the following components:



Because,  $\sim$ , the environment structure $(Q : M : \mathbb{U} \to \mathbb{C} \to \mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C}), ⏚)$ has purification.

□

The following are few concrete examples of environment structures:

- Consider the MUC, $\mathbb{R} \subset \mathbb{C}$. Then,

$$(\mathbb{R} \xrightarrow{Q} \mathsf{CP}^\infty(\mathbb{R} \subset \mathbb{C}), ⏚_r : r \to 1)$$

  is an environment structure where, $⏚_r := (=, 1/r) : r \to 1$

- Consider the MUC, $\mathsf{Mat}_\mathbb{C} \to \mathsf{FMat}(\mathbb{C})$. Then,

$$\mathsf{Mat}_\mathbb{C} \xrightarrow{Q} \mathsf{CP}^\infty(\mathsf{Mat}_\mathbb{C} \subset \mathsf{FMat}(\mathbb{C}))$$

  is an environment structure where, $⏚_{\mathbb{C}^n} : \mathbb{C}^n \to \mathbb{C}; \rho \mapsto Tr(\rho)$.

4.2 Axiomatizing the CP^∞ construction

We are now ready to show that any two environment structures on a MUC are isomorphic if they have purification. Thus, using environment structures one can capture the general structure of the categories of form $\mathsf{CP}^\infty(M : \mathbb{U} \to \mathbb{C})$.

Definition 4.4. *Let $M : \mathbb{U} \to \mathbb{C}$ be a mixed unitary category. Define category* $\mathsf{Env}(M : \mathbb{U} \to \mathbb{C})$ *as follows:*

Objects: *Environment structures $(D : \mathbb{C} \to \mathbb{D}, ⏚)$ for $M : \mathbb{U} \to \mathbb{C}$*

Arrows: $(D : \mathbb{C} \to \mathbb{Y}, ⏚) \xrightarrow{F} (D' : \mathbb{C} \to \mathbb{D}', ⏚)$ *such that*

- *$F : \mathbb{D} \to \mathbb{D}'$ is a strict isomix functor*

- $DF = D'$
- $F(\downarrow) = \downarrow$

Identity arrows: *Identity functor*

Composition: *Linear functor composition*

Lemma 4.5. *Let $M : \mathbb{U} \longrightarrow \mathbb{C}$ be a mixed unitary category. Suppose $(D : \mathbb{C} \longrightarrow \mathbb{D}, \downarrow)$ is an environment structure with purification. Then, $(D : \mathbb{C} \longrightarrow \mathbb{D}, \downarrow)$ is initial in $\mathsf{Env}(\mathbb{C})$.*

For proof, refer Appendix C.

Corollary 4.6. *Suppose $(D : \mathbb{C} \longrightarrow \mathbb{D}, \downarrow)$ is an environment structure with purification for mixed unitary category $M : \mathbb{U} \longrightarrow \mathbb{C}$. Then, $\mathbb{D} \simeq \mathsf{CP}^\infty(M : \mathbb{U} \longrightarrow \mathbb{C})$.*

Proof. By Lemma C.1, $(D : \mathbb{C} \longrightarrow \mathbb{D}, \downarrow)$ is initial object in $\mathsf{Env}(M : \mathbb{U} \longrightarrow \mathbb{C})$. By Lemma 4.3 and C.1, $(Q : \mathbb{C} \longrightarrow \mathsf{CP}^\infty(M : \mathbb{U} \longrightarrow \mathbb{C}), [((u_\oplus^R)^{-1}, U)])$ is an environment structure for $M : \mathbb{U} \longrightarrow \mathbb{C}$ which has purification. Hence, $(Q : \mathbb{X} \longrightarrow \mathsf{CP}^\infty(\mathbb{X}), [((u_\oplus^R)^{-1}, U)])$ is also an initial object in $\mathsf{Env}(M : \mathbb{U} \longrightarrow \mathbb{C})$. Since, initial objects of a category are isomorphic, there exists a $(D : \mathbb{X} \longrightarrow \mathbb{Y}, \downarrow) \xrightarrow{F} (Q : \mathbb{X} \longrightarrow \mathsf{CP}^\infty(\mathbb{X}), [((u_\oplus^R)^{-1}, U)])$ such that $\mathbb{Y} \xrightarrow{F} \mathsf{CP}^\infty(\mathbb{X})$ is full and faithful. □

Acknowledgement

The authors would like to thank Cole Comfort and Jean Simon Lemay for many fruitful discussions and their insights to the project.

References

[1] Samson Abramsky and Chris Heunen. H*-algebras and nonunital Frobenius algebras: first steps in infinite-dimensional categorical quantum mechanics. *arXiv preprint arXiv:1011.6123*, 2010.

[2] R. F. Blute, J. R. B. Cockett, and R. A. G. Seely. Feedback for linearly distributive categories: traces and fixpoints. *Journal of Pure and Applied Algebra*, 154(1-3):27–69, 2000.

[3] R. F. Blute, J. R. B. Cockett, R. A. G. Seely, and T. H. Trimble. Natural deduction and coherence for weakly distributive categories. *Journal of Pure and Applied Algebra*, 113(3):229 – 296, 1996.

[4] Robin Cockett, Cole Comfort, and Priyaa Srinivasan. Dagger linear logic for categorical quantum mechanics. *ArXiv e-prints*, September 2018.

[5] Robin Cockett and Robert Seely. Proof theory for full intuitionistic linear logic, bilinear logic, and mix categories. *Theory and Applications of categories*, 3(5):85–131, 1997.

[6] Bob Coecke and Chris Heunen. Pictures of complete positivity in arbitrary dimension. *Information and Computation*, 250:50–58, 2016.

[7] Stefano Gogioso and Fabrizio Genovese. Infinite-dimensional Categorical Quantum Mechanics. may 2016.

[8] Roberto Maieli and Quintijn Puite. Modularity of proof-nets. *Archive for Mathematical Logic*, 44(2):167–193, 2005.

[9] Mathys Rennela, Sam Staton, and Robert Furber. Infinite-dimensionality in quantum foundations: W*-algebras as presheaves over matrix algebras. *arXiv preprint arXiv:1701.00662*, 2017.

[10] Peter Selinger. Dagger compact closed categories and completely positive maps. *Electronic Notes in Theoretical Computer Science*, 170:139–163, 2007.

A Dagger linearly distributive categories and functors

In this section, we present the definition of a mixed unitary category and other basic definitions from [4]. In [4], we required the unitary objects to be inside the core of an isomix category. However, we have dropped this requirement here as we realized it was not necessary. We still require, however, that the mixor of any two unitary objects should be an isomorphism which is always implied if they are in the core.

We assume that the reader is familiar with the definition of linearly distributive categories [5], mix categories, linear adjoints, linearly distributive functors and linear transformations [5], $*$-autonomous categories, and dagger compact closed categories [10], However, we rcall the essential properties of these structures below:

Linearly distributive categories are categories with two monoidal structures $(\otimes, \top, a_\otimes, u^l_\otimes, u^r_\otimes)$ and $(\oplus, \bot, a_\oplus, u^l_\oplus, u^r_\oplus)$ linked by natural transformations called linear distributors:

$$\delta_L : A \otimes (B \oplus C) \to (A \otimes B) \oplus C \qquad \delta_R : (A \oplus B) \otimes C \to A \oplus (B \otimes C)$$

A monoidal category is a LDC in which both the monoidal structures coincide. A $*$-autonomous category is a LDC in which every object has a chosen right and left dual. LDCs provide a categorical semantics for linear logic. Moreover, LDCs are equipped with graphical calculus, refer [2]. A mix category is a LDC with a mix map $\mathsf{m} : \top \to \bot$ satisfying a certain coherence condition. The mix map gives rise to natural transformations $\mathsf{mx} : A \otimes - \to A \oplus -$ called the right mixor and $\mathsf{mx} : - \otimes A \to - \oplus A$ called the right mixor.

A.1 Dagger linearly distributive categories

Definition A.1. *A dagger linearly distributive category is an LDC with a functor* $(\_)^\dagger : \mathbb{X}^{\mathsf{op}} \to \mathbb{X}$ *and natural isomorphisms*

$$\textbf{laxors:}\; A^\dagger \otimes B^\dagger \xrightarrow{\lambda_\otimes} (A \oplus B)^\dagger \qquad A^\dagger \oplus B^\dagger \xrightarrow{\lambda_\oplus} (A \otimes B)^\dagger$$

$$\top \xrightarrow{\lambda_\top} \bot^\dagger \qquad \bot \xrightarrow{\lambda_\bot} \top^\dagger$$

$$\textbf{involutor:}\; A \xrightarrow{\iota} (A^\dagger)^\dagger$$

such that the following coherences hold:

[†-ldc.1] *Interaction of* $\lambda_\otimes, \lambda_\oplus$ *with associators:*

$$\begin{array}{ccc}
A^\dagger \otimes (B^\dagger \otimes C^\dagger) & \xrightarrow{a_\otimes} & (A^\dagger \otimes B^\dagger) \otimes C^\dagger \\
\downarrow{\scriptstyle 1\otimes\lambda_\otimes} & & \downarrow{\scriptstyle \lambda_\otimes \otimes 1} \\
A^\dagger \otimes (B \oplus C)^\dagger & & (A \oplus B)^\dagger \otimes C^\dagger \\
\downarrow{\scriptstyle \lambda_\otimes} & & \downarrow{\scriptstyle \lambda_\otimes} \\
(A \oplus (B \oplus C))^\dagger & \xrightarrow[(a_\oplus^{-1})^\dagger]{} & ((A \oplus B) \oplus C)^\dagger
\end{array}
\qquad
\begin{array}{ccc}
A^\dagger \oplus (B^\dagger \oplus C^\dagger) & \xrightarrow{a_\oplus} & (A^\dagger \oplus B^\dagger) \oplus C^\dagger \\
\downarrow{\scriptstyle 1\oplus\lambda_\oplus} & & \downarrow{\scriptstyle \lambda_\oplus \oplus 1} \\
A^\dagger \oplus (B \otimes C)^\dagger & & (A \otimes B)^\dagger \oplus C^\dagger \\
\downarrow{\scriptstyle \lambda_\oplus} & & \downarrow{\scriptstyle \lambda_\oplus} \\
(A \otimes (B \otimes C))^\dagger & \xrightarrow[(a_\otimes^{-1})^\dagger]{} & ((A \otimes B) \otimes C)^\dagger
\end{array}$$

[†-ldc.2] *Interaction of* $\lambda_\top, \lambda_\bot$ *with unitors:*

$$\begin{array}{ccc}
\top \otimes A^\dagger & \xrightarrow{\lambda_\top \otimes 1} & \bot^\dagger \otimes A^\dagger \\
\downarrow{\scriptstyle u^R_\otimes} & & \downarrow{\scriptstyle \lambda_\otimes} \\
A^\dagger & \xrightarrow[(u^R_\oplus)^\dagger]{} & (\bot \oplus A)^\dagger
\end{array}
\qquad
\begin{array}{ccc}
\bot \oplus A^\dagger & \xrightarrow{\lambda_\bot \oplus 1} & \top^\dagger \oplus A^\dagger \\
\downarrow{\scriptstyle u^R_\oplus} & & \downarrow{\scriptstyle \lambda_\oplus} \\
A^\dagger & \xrightarrow[(u^R_\otimes)^\dagger]{} & (\top \otimes A)^\dagger
\end{array}$$

and two symmetric diagrams for $u^R_\otimes$ and $u^R_\oplus$ must also be satisfied.

[†-ldc.3] *Interaction of $\lambda_\otimes, \lambda_\oplus$ with linear distributors:*

$$\begin{array}{ccc}
A^\dagger \otimes (B^\dagger \oplus C^\dagger) & \xrightarrow{\delta^L} & (A^\dagger \otimes B^\dagger) \oplus C^\dagger \\
{\scriptstyle 1\otimes\lambda_\oplus}\Big\downarrow & & \Big\downarrow{\scriptstyle \lambda_\otimes\oplus 1} \\
A^\dagger \otimes (B \otimes C)^\dagger & & (A \oplus B)^\dagger \oplus C^\dagger \\
{\scriptstyle \lambda_\otimes}\Big\downarrow & & \Big\downarrow{\scriptstyle \lambda_\oplus} \\
(A \oplus (B \otimes C))^\dagger & \xrightarrow[(\delta^R)^\dagger]{} & ((A \oplus B) \otimes C)^\dagger
\end{array}
\qquad
\begin{array}{ccc}
(A^\dagger \oplus B^\dagger) \otimes C^\dagger & \xrightarrow{\delta^R} & A^\dagger \oplus (B^\dagger \otimes C^\dagger) \\
{\scriptstyle \lambda_\oplus\otimes 1}\Big\downarrow & & \Big\downarrow{\scriptstyle 1\oplus\lambda_\otimes} \\
(A \otimes B)^\dagger \otimes C^\dagger & & A^\dagger \oplus (B \oplus C)^\dagger \\
{\scriptstyle \lambda_\otimes}\Big\downarrow & & \Big\downarrow{\scriptstyle \lambda_\oplus} \\
((A \otimes B) \oplus C)^\dagger & \xrightarrow[(\delta^L)^\dagger]{} & (A \otimes (B \oplus C))^\dagger
\end{array}$$

[†-ldc.4] *Interaction of $\iota : A \to A^{\dagger\dagger}$ with $\lambda_\otimes$, $\lambda_\oplus$:*

$$\begin{array}{ccc}
A \oplus B & \xrightarrow{\iota} & ((A \oplus B)^\dagger)^\dagger \\
{\scriptstyle \iota\oplus\iota}\Big\downarrow & & \Big\downarrow{\scriptstyle \lambda_\otimes^\dagger} \\
(A^\dagger)^\dagger \oplus (B^\dagger)^\dagger & \xrightarrow[\lambda_\oplus]{} & (A^\dagger \otimes B^\dagger)^\dagger
\end{array}
\qquad
\begin{array}{ccc}
A \otimes B & \xrightarrow{\iota} & ((A \otimes B)^\dagger)^\dagger \\
{\scriptstyle \iota\otimes\iota}\Big\downarrow & & \Big\downarrow{\scriptstyle \lambda_\oplus^\dagger} \\
(A^\dagger)^\dagger \otimes (B^\dagger)^\dagger & \xrightarrow[\lambda_\otimes]{} & (A^\dagger \oplus B^\dagger)^\dagger
\end{array}$$

[†-ldc.5] *Interaction of $\iota : A \to A^{\dagger\dagger}$ with $\lambda_\top$, $\lambda_\bot$:*

$$\begin{array}{ccc}
\bot & \xrightarrow{\iota} & (\bot^\dagger)^\dagger \\
& {\scriptstyle \lambda_\bot}\searrow & \Big\downarrow{\scriptstyle \lambda_\top^\dagger} \\
& & \top^\dagger
\end{array}
\qquad
\begin{array}{ccc}
\top & \xrightarrow{\iota} & (\top^\dagger)^\dagger \\
& {\scriptstyle \lambda_\top}\searrow & \Big\downarrow{\scriptstyle \lambda_\bot^\dagger} \\
& & \bot^\dagger
\end{array}$$

[†-ldc.6] $\iota_{A^\dagger} = (\iota_A^{-1})^\dagger : A^\dagger \longrightarrow A^{\dagger\dagger\dagger}$

A **symmetric †-LDC** is a †-LDC which is a symmetric LDC and for which the following additional diagrams commute:

[†-ldc.7] Interaction of $\lambda_\otimes, \lambda_\oplus$ with symmetry maps:

$$\begin{array}{ccc}
A^\dagger \otimes B^\dagger & \xrightarrow{\lambda_\otimes} & (A \oplus B)^\dagger \\
{\scriptstyle c_\otimes}\Big\downarrow & & \Big\downarrow{\scriptstyle c_\oplus^\dagger} \\
B^\dagger \otimes A^\dagger & \xrightarrow[\lambda_\otimes]{} & (B \oplus A)^\dagger
\end{array}
\qquad
\begin{array}{ccc}
A^\dagger \oplus B^\dagger & \xrightarrow{\lambda_\oplus} & (A \otimes B)^\dagger \\
{\scriptstyle c_\oplus}\Big\downarrow & & \Big\downarrow{\scriptstyle c_\otimes^\dagger} \\
B^\dagger \oplus A^\dagger & \xrightarrow[\lambda_\oplus]{} & (B \otimes A)^\dagger
\end{array}$$

Definition A.2. *A* **†-mix category** *is a †-LDC with a mix map, where additionally, the following diagram commutes:*

$$\textbf{[†-mix]} \qquad
\begin{array}{ccc}
\bot & \xrightarrow{\mathsf{m}} & \top \\
{\scriptstyle \lambda_\bot}\Big\downarrow & & \Big\downarrow{\scriptstyle \lambda_\top} \\
\top^\dagger & \xrightarrow[\mathsf{m}^\dagger]{} & \bot^\dagger
\end{array}$$

If m is an isomorphism, then $\mathbb{X}$ is a **†-isomix category**.

Lemma A.3. *Suppose $\mathbb{X}$ is a †-mix category then the following diagram commutes:*

$$\begin{array}{ccc} A^\dagger \oplus B^\dagger & \xrightarrow{\mathsf{mx}} & A^\dagger \otimes B^\dagger \\ {\scriptstyle \lambda_\oplus}\downarrow & & \downarrow{\scriptstyle \lambda_\otimes} \\ (A \otimes B)^\dagger & \xrightarrow[\mathsf{mx}^\dagger]{} & (A \oplus B)^\dagger \end{array}$$

Proof. The proof follows directly from Lemma A.7. □

Lemma A.4. *Let $\mathbb{X}$ be †-LDC. If $(\eta, \varepsilon) : A \dashv\!\!\vdash B$ then $(\lambda_\top \varepsilon^\dagger \lambda_\oplus^{-1}, \lambda_\otimes \eta^\dagger \lambda_\perp^{-1}) : B^\dagger \dashv\!\!\vdash A^\dagger$.*

A.2 Dagger linear functors

Having defined †-isomix categories, we may now describe the appropirate functors between these categories. At a fundamental level, one would expect such functors to preseve the linear structure and the dagger.

Definition A.5. *A* **Frobenius functor** *is a linear functor F such that:*

[FLF.1] $F_\otimes = F_\oplus$

[FLF.2] $m_\otimes = \nu_\oplus^R = \nu_\oplus^L$

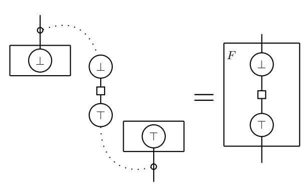
[FLF.3] $n_\oplus = \nu_\otimes^L = \nu_\otimes^R$

Observe that it follows by definition that for any †-LDC $\mathbb{X}$, $(\_)^\dagger : \mathbb{X}^{\mathsf{op}} \longrightarrow \mathbb{X}$ is a Frobenius functor.

Definition A.6. *Suppose $\mathbb{X}$ and $\mathbb{Y}$ are mix categories. $F : \mathbb{X} \longrightarrow \mathbb{Y}$ is a* **mix functor** *if it is a Frobenius functor such that:*

$$\textbf{[mix-FF]} \qquad F(\perp) \xrightarrow{n_\perp} \perp \xrightarrow{\mathsf{m}} \top \xrightarrow{m_\top} F(\top) \quad = \quad F(\perp) \xrightarrow{F(\mathsf{m})} F(\top)$$

This is diagrammatically represented using functor boxes as follows:

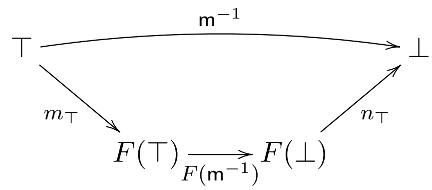


Lemma A.7. *Mix functors preserve the mix map:*

$$\begin{array}{ccc} F(A) \otimes F(B) & \xrightarrow{\mathsf{mx}} & F(A) \oplus F(B) \\ {\scriptstyle m_\otimes}\downarrow & & \uparrow{\scriptstyle n_\oplus} \\ F(A \otimes B) & \xrightarrow[F(\mathsf{mx})]{} & F(A \oplus B) \end{array}$$

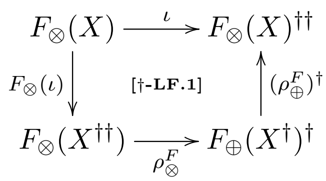
Definition A.8. *A Frobenius functor between isomix categories is an* **isomix functor** *in case it is a mix functor which satisfies, in addition, the following diagram:*

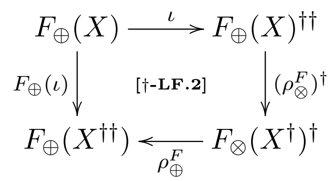
[isomix-FF]



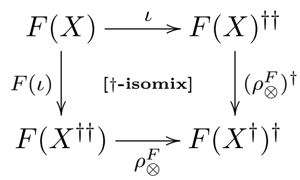
Note that when $\mathbb{X}$ is a †-isomix category, $(\_)^\dagger : \mathbb{X}^{\mathsf{op}} \to \mathbb{X}$ is an isomix functor.

Definition A.9. $F : \mathbb{X} \to \mathbb{Y}$ *is a* **†-linear functor** *between †-LDCs when it is a linear functor equipped with a linear natural isomorphism* $\rho^F = (\rho^F_\otimes : F_\otimes(A^\dagger) \to F_\oplus(A)^\dagger, \rho^F_\oplus : F_\otimes(A)^\dagger \to F_\oplus(A^\dagger))$ *called the* **preservator***, such that the following diagrams commute:*





Observe that when F is a mix functor between †-isomix categories, then $F_\otimes = F_\oplus$ and the preservators become pairwise inverses: $\rho^F_\otimes = (\rho^F_\oplus)^{-1}$. This means the squares [**†-LF.1**] and [**†-LF.2**] coincide to give a single condition for the tensor preservator:



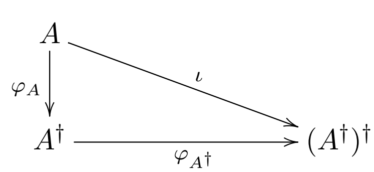
A.3 Unitary structure

The notion of unitary maps is central to both quantum information theory as well as quantum mechanics since the evolution of a closed quantum system is described by such maps. Categorically, within a †-category, a unitary map is an isomorphism $f : A \to B$ such that $f^{-1} = f^\dagger$. This definiton of unitary isomorphism cannot be used directly within the framework of †-LDCs since the types of $f^{-1} : B \to A$ and $f^\dagger : B^\dagger \to A^\dagger$ are different. It is therefore apparent that one can only ask to have unitary isomorphisms between certain objects, which we call "unitary objects":

Definition A.10. *An object A in a †-isomix category, $\mathbb{X}$, is a* **unitary object** *if it comes equipped with an isomorphism, $\varphi_A : A \to A^\dagger$, called the* **unitary structure map** *of A, such that:*

[C.1] *U is closed to $(\_)^\dagger$ so that* $\varphi_{A^\dagger} = ((\varphi_A)^{-1})^\dagger$

[C.2] *the following diagram commutes:*



We can now define what it means for a isomorphism to be unitary:

Definition A.11. *Suppose A and B are unitary objects. An isomorphism $A \xrightarrow{f} B$ is said to be a* **unitary isomorphism** *if the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{\varphi_A} & A^\dagger \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle f^\dagger} \\ B & \xrightarrow[\varphi_B]{} & B^\dagger \end{array}$$

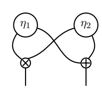
Observe that φ is "twisted" natural for all unitary isomorphisms, thus, unitary isomorphisms compose and contain the identity maps. In a category in which the unitary structure maps are identity morphisms, one recovers the usual notion of unitary isomorphisms.

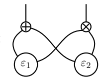
Often we shall want the unitary objects to have linear adjoints (or duals) but we shall need the analogue of †-duals from categorical quantum mechanics:

Definition A.12. *A* **unitary linear duality** $(\eta, \varepsilon) : A \dashv\!\!\vdash_u B$ *between unitary objects A and B is a linear duality satisfying in addition:*

[**Udual**.]
$$\begin{array}{ccc} \top & \xrightarrow{\eta_A} & A \oplus B \\ {\scriptstyle \lambda_\top}\downarrow & & \downarrow{\scriptstyle \varphi_A \oplus \varphi_B} \\ \bot^\dagger & (a) & A^\dagger \oplus B^\dagger \\ {\scriptstyle \varepsilon^\dagger}\downarrow & & \downarrow{\scriptstyle c_\oplus} \\ (B \otimes A)^\dagger & \xrightarrow[\lambda_\oplus^{-1}]{} & B^\dagger \oplus A^\dagger \end{array} \qquad \begin{array}{ccc} A \otimes B & \xrightarrow{\varphi_A \otimes \varphi_B} & A^\dagger \otimes B^\dagger \\ {\scriptstyle c_\otimes}\downarrow & & \downarrow{\scriptstyle \lambda_\otimes} \\ B \otimes A & (b) & (A \oplus B)^\dagger \\ {\scriptstyle \varepsilon_A}\downarrow & & \downarrow{\scriptstyle \eta_A^\dagger} \\ \bot & \xrightarrow[\lambda_\bot]{} & \top^\dagger \end{array}$$

Lemma A.13. *Suppose $(\eta_1, \varepsilon_1) : V_1 \dashv\!\!\vdash_u U_1$ and $(\eta_2, \varepsilon_2) : V_2 \dashv\!\!\vdash_u U_2$. Then, $(V_1 \otimes V_2) \dashv\!\!\vdash_u (U_1 \oplus U_2)$.*

Proof. Define $(\eta', \varepsilon') : (V_1 \otimes V_2) \dashv\!\!\vdash_u (U_1 \oplus U_2)$ where $\eta' =$ 

$\varepsilon' =$ 

then this is easily checked to be a unitary linear adjoint. □

A.4 Mixed unitary categories

With the definition of unitary objects in place, one could ask for a †-isomix category in which all the objects are unitary; or a †-isomix category with a full sub †-isomix category of unitary objects. The former notion is formalized by so called unitary categories, generalising †-monoidal categories; the latter is formalized by so called mixed unitary categories.

Definition A.14. *A* **unitary category** *is a †-isomix category, $\mathbb{U}$, such that*

[U.1] *Every object $U \in \mathbb{U}$ is a unitary object*

[U.2] *The unitary structure maps interact coherently with the mix map and the unit laxors:*

$$\bot \xrightarrow[\simeq]{\varphi_\bot} \bot^\dagger \xrightarrow[\simeq]{\lambda_\top^{-1}} \top \;=\; \bot \xrightarrow{\mathsf{m}} \top \qquad \bot \xrightarrow[\simeq]{\lambda_\top} \top^\dagger \xrightarrow[\simeq]{\varphi_\top^{-1}} \top \;=\; \bot \xrightarrow{\mathsf{m}} \top$$

$$\top^\dagger \xrightarrow[\simeq]{\varphi_\top^{-1}} \bot^\dagger \xrightarrow[\simeq]{\lambda_\top} \bot^\dagger \;=\; \top^\dagger \xrightarrow{\mathsf{m}^\dagger} \bot^\dagger \qquad \top^\dagger \xrightarrow[\simeq]{\lambda_\bot^{-1}} \bot \xrightarrow[\simeq]{\varphi_\bot} \bot^\dagger \;=\; \top^\dagger \xrightarrow{\mathsf{m}^\dagger} \bot^\dagger$$

[U.3] *The unitary structure maps interact coherently with the mixor and the monoidal laxors:*

$$A \otimes B \xrightarrow[\simeq]{\varphi_A \otimes \varphi_B} A^\dagger \otimes B^\dagger \xrightarrow[\simeq]{\lambda_\oplus} (A \oplus B)^\dagger \xrightarrow[\simeq]{\varphi_{A \oplus B}^{-1}} A \oplus B \qquad = \qquad A \otimes B \xrightarrow{\mathsf{mx}} A \oplus B$$

$$A \otimes B \xrightarrow[\simeq]{\varphi_{A \otimes B}} (A \otimes B)^\dagger \xrightarrow[\simeq]{\lambda_\otimes^{-1}} A^\dagger \oplus B^\dagger \xrightarrow[\simeq]{\varphi_A \oplus \varphi_B} A \oplus B \qquad = \qquad A \otimes B \xrightarrow{\mathsf{mx}} A \oplus B$$

A unitary category is a compact LDC by defintion. A †-monoidal category is a strict unitary category in which the unitary structure map and the mix map are identity morphisms. Similarily, a †-compact closed category is a strict unitary category in which all objects have unitary duals.

We now turn to the the definition of a mixed unitary category:

Definition A.15. *A* **mixed unitary category (MUC)***, ${}_{\mathbb{U}}\mathbb{C}_M$, is a †-isomix category, $\mathbb{C}$, equipped with a †-isomix functor $M : \mathbb{U} \longrightarrow \mathbb{C}$ from a unitary category $\mathbb{U}$ to $\mathbb{C}$. If ${}_{\mathbb{U}}\mathbb{C}_M$ is an MUC where $\mathbb{C}$ is a $*$-autonomous category, then ${}_{\mathbb{U}}\mathbb{C}_M$ is said to be a* **$*$-MUC***. If moreover every object in $\mathbb{U}$ has unitary duals, ${}_{\mathbb{U}}\mathbb{C}_M$ is said to be a* **$*$-mixed unitary category with duals** *($*$-MUdC).*

All dagger monoidal categories together with the identity functor are mixed unitary categories. The inclusion of a full subcategory of unitary objects of a †-isomix category with unitary structure is a MUC.

A source of toy examples of MUCs are given by Abelian groups with involution:

Definition A.16. *A* **group with involution** *is a group $(G, \circ, e)$ together with a function $\overline{(\_)} : G \longrightarrow G$ such that,*

Involution Law *for all $g \in G$, $\overline{\overline{g}} = g$.*

Antihomorphism law *for all $g, h \in G$, $\overline{g \circ h} = \overline{h} \circ \overline{g}$.*

Preservation of identity $\overline{e} = e$

The group of complex numbers has ab involution given by the complex conjugation. The group of $n \times n$ matrices over $\mathbb{R}$ has an involution given by transposition. The polynomial ring $R[x]$ over a commutative ring R has an involution given by taking $P(x) \mapsto P(-x)$.

Let $(G, \circ, e)$ be an Abelian group with involution. Consider the discrete category $\mathbb{X} := \mathbb{D}(G, \circ, e)$ for which the objects are elements of the group $(G, ., e)$. This category is obviously a monoidal category with tensor given by group multiplication and tensor unit given by the unit of the group multiplication. $\mathbb{X}$ is moreover a compact closed category: for all $g \in G$, $g^* := g^{-1}$ with unit $(e = g^{-1} \circ g)$ and counit $(g \circ g^{-1} = e)$ morphisms. It is also a conjugative monoidal category with the conjugation functor given by $\overline{(\_)} : \mathbb{X}^{\mathsf{rev}} \longrightarrow \mathbb{X}$ given by the group involution: $g \mapsto \overline{g}$. Cutely, $\mathbb{X}$ is a compact † isomix $*$-autonomous category with $g^\dagger := \overline{g^*}$. The unitary objects are those group elements such that $\overline{g^{-1}} = g$. Note that this model is a compact LDC.

Consider the concrete discrete category $\mathbb{D}(\mathbb{C}, +, 1)$. In this category, for all complex numbers $a + ib$, $(a + ib)^* := -a - ib$ and $\overline{a + ib} := a - ib$. Hence,

$$(a + ib)^\dagger := \overline{(a + ib)^*} = \overline{(-a - ib)} = -a + ib$$

The preunitary objects in this category are precisely complex numbers of form ib.

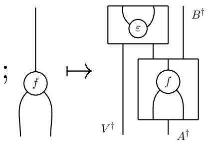
Prominent examples of MUCs are given by the inclusion functor from the core. Markedly: the category of finite-dimensional framed vector spaces and linear maps, finiteness modules, and the category of Chu spaces of complex vector spaces over $\mathbb{X}$ give MUCs in this manner [4].

B CP^∞ construction

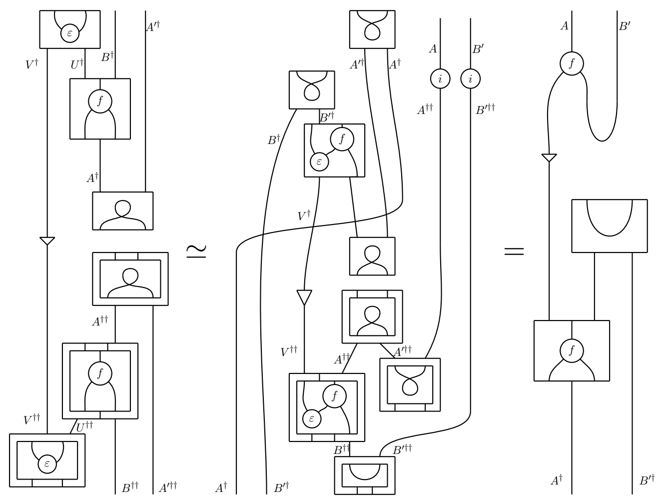
Lemma B.1. *If $M : \mathbb{U} \longrightarrow \mathbb{C}$ is a $*$-MUdC then $\mathsf{CP}^\infty(M : \mathbb{U} \longrightarrow \mathbb{C})$ is also a $*$-MUdC.*

Proof. (Sketch) We first oberve that $\mathsf{CP}^\infty({}_{\mathbb{U}}\mathbb{C}_M)$ is a iso-mix †-LDC. We have already proven that $\mathsf{CP}^\infty({}_{\mathbb{U}}\mathbb{C}_M)$ is an isomix category. Suppose $\mathbb{X}$ is a $*$-MUdC, then the † functor for $\mathsf{CP}^\infty({}_{\mathbb{U}}\mathbb{C}_M)$ is defined as follows: Suppose $f : A \xrightarrow[U]{} B$ and $(\eta, \varepsilon) : V \dashv\!\!\vdash_u U$ with $A' \dashv\!\!\vdash A$ and $B' \dashv\!\!\vdash B \in \mathbb{C}$ then,

$$\dagger : \mathsf{CP}^\infty({}_{\mathbb{U}}\mathbb{C}_M)^{\mathrm{op}} \longrightarrow \mathsf{CP}^\infty({}_{\mathbb{U}}\mathbb{C}_M);$$



We need to prove that † is well-defined: $f \sim g \Rightarrow f^\dagger \sim g^\dagger$.



The equality is proved by using the snake diagrams and **[C.1]**, **[C.2]** and **[Udual.]**.

Suppose $f : A \to U_1 \oplus B$ and $g : B \longrightarrow U_2 \oplus C$ with $(\eta_1, \varepsilon_1) : U_1 \dashv\!\!\vdash_u V_1$ and $(\eta_2, \varepsilon_2) : U_2 \dashv\!\!\vdash_u V_2$, then † preserves composition, that is $(fg)^\dagger = g^\dagger f^\dagger$:

$$\frac{(fg) : A \longrightarrow (U_1 \oplus U_2) \oplus C}{(fg)^\dagger : C^\dagger \to (V_1 \otimes V_2)^\dagger \oplus A^\dagger}$$

$$\frac{g^\dagger : C^\dagger \longrightarrow U_2^\dagger \oplus B^\dagger \qquad\qquad f^\dagger : B^\dagger \longrightarrow U_1^\dagger \oplus A^\dagger}{(g^\dagger f^\dagger) : C^\dagger \to (V_2^\dagger \otimes V_1^\dagger) \oplus A^\dagger}$$

To prove that $(fg)^\dagger = (g^\dagger f^\dagger)$ in $\mathsf{CP}^\infty({}_{\mathbb{U}}\mathbb{C}_M)$, represent the maps in circuit calculus and fuse the †-boxes. Once the †-boxes are fused, use Lemma 3.2 to show that both Kraus operations belong to the same equivalence. † preserves identity map since $((u_\oplus^R)^{-1}, u_\oplus^L) : \top \dashv\!\!\vdash_u \bot$. Hence, † is a functor.

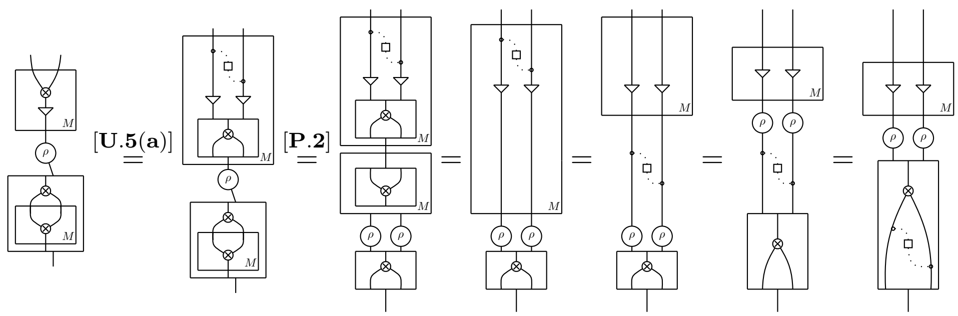
All the basic natural isomorphisms associated with † functor - $\lambda_\oplus, \lambda_\otimes, \lambda_\bot, \lambda_\top, \iota$ - are lifted from $\mathbb{C}$ using $Q : \mathbb{C} \hookrightarrow \mathsf{CP}^\infty({}_{\mathbb{U}}\mathbb{C}_M)$ which is defined in Lemma 3.7. The lifted morphisms are natural in $\mathsf{CP}^\infty({}_{\mathbb{U}}\mathbb{C}_M)$ since their ancillaries are unitarily isomorphic. Since † is functorial, all commuting diagrams are preserved. By the same argument, unitary structure is preserved under Q.

Thus, $\mathsf{CP}^\infty({}_{\mathbb{U}}\mathbb{C}_M)$ is a mixed unitary category: as Q preserves all unitary linear adjoints this makes $\mathsf{CP}^\infty({}_{\mathbb{U}}\mathbb{C}_M)$ a $*$-MUdC. □

Lemma B.2. *The following diagarms commute:*

$$
\begin{array}{ccc}
M(C)\otimes M(D) & \xrightarrow{M(\varphi)\otimes M(\varphi)} & M(C^\dagger)\otimes M(D^\dagger) \\
\downarrow{\scriptstyle m_\otimes^M} & & \downarrow{\scriptstyle \rho\otimes\rho} \\
M(C\otimes D) & & M(C)^\dagger\otimes M(D)^\dagger \\
\downarrow{\scriptstyle M(\varphi)} & \textbf{(a)} & \downarrow{\scriptstyle \lambda_\otimes} \\
M((C\otimes D)^\dagger) & & (M(C)\oplus M(D))^\dagger \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \mathsf{mx}^\dagger} \\
(M(C)\otimes M(D))^\dagger & \xrightarrow[(m_\otimes^M)^\dagger]{} & (M(C)\otimes M(D))^\dagger
\end{array}
\qquad
\begin{array}{ccc}
M(C)\oplus M(D) & \xrightarrow{M(\varphi_C)\oplus M(\varphi_D)} & M(C^\dagger)\oplus M(D^\dagger) \\
\downarrow{\scriptstyle n_\oplus^{-1}} & & \downarrow{\scriptstyle \rho\oplus\rho} \\
M(C\oplus D) & & M(C)^\dagger\oplus M(D)^\dagger \\
\downarrow{\scriptstyle M(\varphi)} & \textbf{(b)} & \downarrow{\scriptstyle \lambda_\oplus} \\
M((C\oplus D)^\dagger) & & (M(C)\otimes M(D))^\dagger \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle (\mathsf{mx}^{-1})^\dagger} \\
(M(C\oplus D))^\dagger & \xleftarrow[n_\oplus^\dagger]{} & (M(C)\oplus M(D))^\dagger
\end{array}
$$

Proof.



The commuting diagram (b) is proved similarly.

□

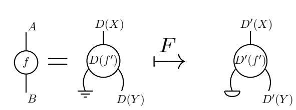
C Environment structures

Lemma C.1. *Suppose* $(D :\ _{\mathbb{U}}\mathbb{C}_M \longrightarrow \mathbb{D}, ⏚)$ *is an environment structure with purification. Then,* $(D :\ _{\mathbb{U}}\mathbb{C}_M \longrightarrow \mathbb{D}, ⏚)$ *is initial in* $\mathsf{Env}(\mathbb{C})$.

Proof. Suppose $(D' : \mathbb{X} \to \mathbb{Y}', ⌓) \in \mathsf{Env}(\mathbb{C})$. We show that there is a unique strict isomix functor $F : \mathbb{Y} \to \mathbb{Y}'$ such that $DF = D'$ and $F(⏚) = ⌓$.

Define $F : \mathbb{D} \to \mathbb{D}'$ as follows:

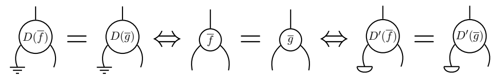
- Since $(D, ⏚)$ has purification, $D : \mathbb{C} \longrightarrow \mathbb{D}$ is bijective on objects. Then for all $A \in \mathbb{D}$, $A = D(X)$ for a unique $X \in \mathbb{C}$. Then,
$$F(A) := D'(X)$$
- Let $f : A \to B \in \mathbb{D}$. Since $(D : \mathbb{C} \longrightarrow \mathbb{D}, ⏚)$ has purification,



where $F(⏚) = ⌓$.

This fixes the definition of F. To prove that F is well-defined on arrows we need to show that $f = g \Rightarrow F(f) = F(g)$. Since, $(D, \downarrow)$ has purification, let

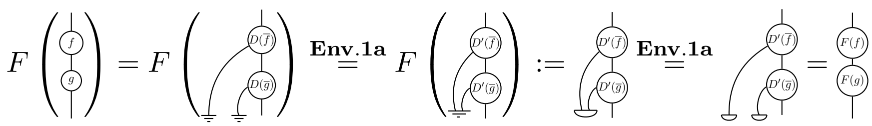
Then,



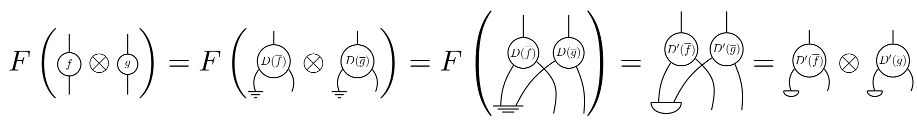
$F : \mathbb{D} \longrightarrow \mathbb{D}'$ preserves identity:

$$F(1_A) = F(1_{D(X)}) = F(D(1_X)) = D'(1_X) = 1_{D'(X)} = 1_{F(D(X))} = 1_{F(A)}$$

$F : \mathbb{D} \longrightarrow \mathbb{D}'$ preserves compostion:



$F : \mathbb{D} \longrightarrow \mathbb{D}'$ is strict monoidal in $\otimes$:



and, $F((u_\otimes^L)_A) = F((u_\otimes^L)_{D(X)}) = F(D((u_\otimes^L)_X)) = D'((u_\otimes^L)_X) = (u_\otimes^L)_{D'(X)} = (u_\otimes^L)_{F(D(X))} = (u_\otimes^L)_{F(A)}$

Smiliarly, it can be proved that F is strict comonoidal in $\oplus$.

Define $F_\otimes = F_\oplus := F$ and linear strengths to be identity maps. Thus, F is a unique strict Frobenius functor. F is an isomix functor because D and D' preserve the mix map m on the nose. $\square$