Monotones in General Resource Theories

Extended Abstract

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1 Introduction

Taking a pragmatic perspective on certain physical phenomena wherein one focuses on how different processes constitute resources dates back to the study of heat engines and the advent of thermodynamics. One asks, for example, what work can be achieved given access to a heat bath and states out of thermal equilibrium, such as a compressed gas.

Soon enough, it was understood that resources of thermal nonequilibrium could be of informational character. One of the most famous examples is the Szilard's engine [1], which uses information about the state of a system in order to perform work. Resource-theoretic thinking has expanded with the development of information theory. The pioneering work of Claude Shannon [2] is centered around questions regarding the convertibility of communication resources.

The second notable place after thermodynamics where resource-theoretic ideas enter the realm of theoretical physics therefore comes with the rising prominence of using information theory in physics. Out of these efforts, the most relevant one for us is the development of quantum information theory. Especially the study of quantum entanglement has led to a multitude of results [3, 4], for which an explicit mathematical framework [5] underlying the resource aspect proved to be useful, as opposed to just the resource-theoretic mindset being present implicitly. Many other examples of the use of resource theories within quantum information theory followed [6], calling for a general investigation of the structure of such theories.

Two examples of abstract frameworks for resource theories are partitioned process theories [7] and ordered commutative monoids [8]. The first one focuses on the notion of free operations and the conversion relations between resources are defined in terms of this notion. On the other hand, ordered commutative monoids formalize the resource ordering itself and can be used to study its properties. In order to be able to get the advantages of both of these approaches, we use a framework that contains partitioned process theories and ordered commutative monoids as special cases. However, a detailed presentation of this framework is postponed to a follow-up article [9]. Here, we only give a brief overview of its contents.

2 A Framework for Resource Theories

The framework makes no distinction a priori between resources as states and resources as channels. We therefore consider a set of resources $\mathcal{R}$, which may include both states and channels. Instead of the resource composition being prescribed by wirings of states and processes as in the partitioned process theory framework, we use an explicit map $\star$ that describes the composition of resources. However, there might be multiple ways of composing two resources $r, s \in \mathcal{R}$. The object $r \star s$ is therefore not a resource, but a set of resources $r \star s \in \mathcal{P}(\mathcal{R})$, where $\mathcal{P}(\mathcal{R})$ denotes the power set of $\mathcal{R}$.

The interpretation of the two operations $\star$ and $\cup$ is such that $\star$ represents a conjunction of resources while the union represents disjunction. That is, $r \star s$ means that the agent has both r and s (and they can be combined). On the other hand, the set $\{t, u\} \subseteq \mathcal{R}$ represents an agent having either resource t or resource u, but not both. Therefore, we require that the $\star$ operation distributes across unions, just like conjunction distributes across disjunction. That is, for any $S, T \in \mathcal{P}(\mathcal{R})$, we have

$$S \star T = \bigcup_{s \in S, t \in T} s \star t. \tag{1}$$

Combining these gives us a commutative monoid $(\mathcal{P}(\mathcal{R}), \star)$ with $\star$ that distributes across unions. In a resource theory, we presume that there is a distinguished subset of states and channels that are *free* in the sense that one can access them in unlimited supply. This subset is denoted by $\mathcal{R}_{\text{free}} \subseteq \mathcal{R}$. The conversion relation between resources—both states and channels—is then induced by this choice. The allowed conversions are those that arise via a composition with elements of the free set $\mathcal{R}_{\text{free}}$.

Definition 1. A **resource theory** $(\mathcal{R}, \mathcal{R}_{\text{free}}, \star)$ consists of

1. A set of resources $\mathcal{R}$,
2. a commutative, monoidal operation $\star \colon \mathcal{P}(\mathcal{R}) \times \mathcal{P}(\mathcal{R}) \to \mathcal{P}(\mathcal{R})$ that distributes across unions, and
3. a subset of free resources $\mathcal{R}_{\text{free}} \subseteq \mathcal{R}$, such that $(\mathcal{P}(\mathcal{R}_{\text{free}}), \star)$ is a submonoid of $(\mathcal{P}(\mathcal{R}), \star)$.

Since $(\mathcal{P}(\mathcal{R}), \star)$ is a monoid, there is a *neutral* set of resources, denoted by 0, that satisfies

$$0 \star S = S = S \star 0 \tag{2}$$

for all $S \in \mathcal{P}(\mathcal{R})$. It differs from the empty set $\emptyset \in \mathcal{P}(\mathcal{R})$, which satisfies

$$\emptyset \star S = \emptyset = S \star \emptyset. \tag{3}$$

If $r \star s = \emptyset$, the interpretation is that the two resources $r \star s$ are mutually incompatible—there is no way to combine them. On the other hand, if $r \star s \subseteq 0$, then any way to combine r and s produces a resource in the neutral set, thus effectively discarding them since elements of the neutral set cannot be used for non-trivial conversions.

The fact that $(\mathcal{P}(\mathcal{R}_{\text{free}}), \star)$ is a submonoid of $(\mathcal{P}(\mathcal{R}), \star)$ means that $0 \in \mathcal{R}_{\text{free}}$ and that $\mathcal{R}_{\text{free}}$ is closed under $\star$; i.e.,

$$\mathcal{R}_{\text{free}} \star \mathcal{R}_{\text{free}} = \mathcal{R}_{\text{free}}. \tag{4}$$

This captures the idea that combining free resources together cannot produce a non-free resource.

Given a resource theory $(\mathcal{R}, \mathcal{R}_{\text{free}}, \star)$, we can define the ordering of resources $\succeq$ by

$$r \succeq s \quad \iff \quad s \in \mathcal{R}_{\text{free}} \star r \tag{5}$$

for any $r, s \in \mathcal{R}$, where $\mathcal{R}_{\text{free}} \star r$ is a shorthand notation for $\mathcal{R}_{\text{free}} \star \{r\}$. The ordering relation captures whether r can be converted to s by means of composition with the free resources. It can be used to determine the value of resources with respect to the choice of the partition of $\mathcal{R}$ into free and non-free resources. If $r \succeq s$, then we would say that r is better than (or equivalent to) s as a resource in the resource theory $(\mathcal{R}, \mathcal{R}_{\text{free}}, \star)$. With this ordering, the set of resources becomes a preordered set $(\mathcal{R}, \succeq)$.

Similarly, we can define the ordering of sets of resources by

$$S \succeq T \quad \iff \quad T \subseteq \mathcal{R}_{\text{free}} \star S, \tag{6}$$

for any $S, T \in \mathcal{P}(\mathcal{R})$. Again, $(\mathcal{P}(\mathcal{R}), \succeq)$ is a preordered set.

3 Useful Order-Theoretic Notions

The natural maps between ordered sets are the order-preserving functions.

Definition 2. Let $(\mathcal{A}, \succeq)$ and $(\mathcal{B}, \succeq')$ be two arbitrary preordered sets. A function $f \colon (\mathcal{A}, \succeq) \to (\mathcal{B}, \succeq')$ is **order-preserving** if

$$a_1 \succeq a_1 \implies f(a_1) \succeq' f(a_2) \tag{7}$$

holds for all $a_1, a_2 \in \mathcal{A}$.

We can use order-preserving functions to learn about $(\mathcal{R}, \succeq)$. One of the most common practices is to find so-called *resource monotones*: order-preserving maps from $(\mathcal{R}, \succeq)$ to the totally ordered set of extended real numbers $(\overline{\mathbb{R}}, \geq)$.

Definition 3. Let $(\mathcal{R}, \mathcal{R}_{\text{free}}, \star)$ be a resource theory. A **resource monotone** (or monotone for short) is a function $f\colon \mathcal{R} \to \overline{\mathbb{R}}$ such that for all $r, s \in \mathcal{R}$, we have

$$s \in \mathcal{R}_{\text{free}} \star r \implies f(r) \geq f(s). \tag{8}$$

For the resource preorder $(\mathcal{R}, \succeq)$ defined by (5), condition (8) can be expressed as $r \succeq s \implies f(r) \geq f(s)$, so that such an f is indeed an order-preserving map from $(\mathcal{R}, \succeq)$ to $(\overline{\mathbb{R}}, \geq)$.

Monotones are the main objects of study of this extended abstract. Their significance is manifold. Firstly, since they contain information about the convertibility of resources, they can help answer whether a particular resource can be converted to another one. A suitable set of monotones provides an efficient characterization of the ordering of pairs of resources. Secondly, they allow us to derive global and local properties of the ordering relation $\succeq$ itself. Finally, monotones can often have operational significance in quantifying the degree of success in some operationally defined task.

In this work, we study the ways in which resource monotones can be constructed. We look at examples of common constructions of monotones appearing in the literature on resource theories and identify more general procedures, which they are instances of. This helps us organize various monotones and understand the connections between them as well as obtain generally applicable methods for generating new interesting monotones in any resource theory of interest. All of the monotone constructions we discuss fall within the following very broad scheme.

Broad Scheme. We find a preordered set $(\mathcal{A}, \succeq_{\mathcal{A}})$ and order-preserving maps σ_1 and σ_2:

$$\sigma_1\colon (\mathcal{R}, \succeq) \to (\mathcal{A}, \succeq_{\mathcal{A}}) \qquad\qquad \sigma_2\colon (\mathcal{A}, \succeq_{\mathcal{A}}) \to (\overline{\mathbb{R}}, \geq) \tag{9}$$

Composing the two order-preserving maps gives a monotone $(\mathcal{R}, \succeq) \to (\overline{\mathbb{R}}, \geq)$. Broadly speaking, the aim of this work is thus to illuminate which choices of $(\mathcal{A}, \succeq_{\mathcal{A}})$, σ_1, and σ_2 lead to monotones that are either prevalent in the literature or interesting for other reasons.

A concept that we will find useful is that of downward and upward closed sets. We make use of these repeatedly.

Definition 4. Let $(\mathcal{R}, \mathcal{R}_{\text{free}}, \star)$ be a resource theory defining a preorder $(\mathcal{R}, \succeq)$. A set $D \subseteq \mathcal{R}$ is **downward closed** if for all $r \in \mathcal{R}$ the implication

$$d \succeq r \text{ for some } d \in D \implies r \in D \tag{10}$$

holds. Equivalently, D is downward closed if $\mathcal{R}_{\text{free}} \star D = D$. The set of all downward closed subsets of $\mathcal{R}$ is denoted by $\mathcal{DC}(\mathcal{R})$. On the other hand, a set of resources U is **upward closed** if for all $r \in \mathcal{R}$ the implication

$$r \succeq u \text{ for some } u \in U \implies r \in U \tag{11}$$

holds. The set of all upward closed subsets of $\mathcal{R}$ is denoted by $\mathcal{UC}(\mathcal{R})$.

Definition 5. We also have a map $\downarrow\colon \mathcal{R} \to \mathcal{DC}(\mathcal{R})$ termed the **free image map** and defined by

$$\downarrow r = \{s \in \mathcal{R} \mid s \in \mathcal{R}_{\text{free}} \star r\} \tag{12}$$

for any $r \in \mathcal{R}$. Similarly, the **free preimage map** $\uparrow\colon \mathcal{R} \to \mathcal{UC}(\mathcal{R})$ is defined by

$$\uparrow r = \{s \in \mathcal{R} \mid r \in \mathcal{R}_{\text{free}} \star s\}. \tag{13}$$

In the language of order theory, we can identify $\downarrow r$ as the *downward closure* of $\{r\}$ and $\uparrow r$ as the *upward closure* of $\{r\}$ with respect to the preordered set $(\mathcal{R}, \succeq)$.

Notice that both $\mathcal{DC}(\mathcal{R})$ and $\mathcal{UC}(\mathcal{R})$ have a natural ordering in terms of subset inclusion which makes $\uparrow$ and $\downarrow$ into order-preserving maps. In particular, we have partially ordered sets $(\mathcal{DC}(\mathcal{R}), \supseteq)$ and $(\mathcal{UC}(\mathcal{R}), \subseteq)$. With this choice, one can show that $\downarrow\colon (\mathcal{R}, \succeq) \to (\mathcal{DC}(\mathcal{R}), \supseteq)$ and $\uparrow\colon (\mathcal{R}, \succeq) \to (\mathcal{UC}(\mathcal{R}), \subseteq)$ are both order-preserving.

Consequently, monotones for the partial orders $(\mathcal{DC}(\mathcal{R}), \supseteq)$ and $(\mathcal{UC}(\mathcal{R}), \subseteq)$ can be pulled back to monotones for $(\mathcal{R}, \succeq)$. We investigate such constructions of resource monotones in the following section, where the role of $(\mathcal{A}, \succeq_{\mathcal{A}})$ in the Broad Scheme is associated with either $(\mathcal{DC}(\mathcal{R}), \supseteq)$ or $(\mathcal{UC}(\mathcal{R}), \subseteq)$.

4 Generalized Resource Yield and Generalized Resource Cost

4.1 f-yield and f-cost Given a Function f Defined on All Resources

Consider any function $f\colon \mathcal{R} \to \overline{\mathbb{R}}$ (not necessarily a monotone), and define two functions f-max and f-min by

$$f\text{-max}\colon \mathcal{DC}(\mathcal{R}) \to \overline{\mathbb{R}} \qquad\qquad f\text{-min}\colon \mathcal{UC}(\mathcal{R}) \to \overline{\mathbb{R}}$$
$$S \mapsto \sup f(S) \qquad\qquad S \mapsto \inf f(S)$$

where $f(S)$ denotes the image of S under f. As a function from the partially ordered set $(\mathcal{DC}(\mathcal{R}), \supseteq)$ to the totally ordered set $(\overline{\mathbb{R}}, \geq)$, f-max is clearly order-preserving. Similarly, f-min is an order-preserving map between $(\mathcal{UC}(\mathcal{R}), \subseteq)$ and $(\overline{\mathbb{R}}, \geq)$.

With the maps $\downarrow$ and $\uparrow$ described in the previous section, we can pull f-max and f-min back to monotones on $\mathcal{R}$. In particular, we get real-valued functions on $\mathcal{R}$ defined by

$$f\text{-yield}(r) := f\text{-max}(\downarrow r) = \sup\{f(s) \mid s \in \downarrow r\} = \sup\{f(s) \mid s \in \mathcal{R}_{\text{free}} \star r\} \tag{14}$$
$$f\text{-cost}(r) := f\text{-min}(\uparrow r) = \inf\{f(s) \mid s \in \uparrow r\} = \inf\{f(s) \mid r \in \mathcal{R}_{\text{free}} \star s\}, \tag{15}$$

which are both resource monotones on $(\mathcal{R}, \succeq)$. The f-yield of r is the largest value of f among the resources that can be obtained from r for free. On the other hand, the f-cost of r is the smallest value of f among the resources one can use to obtain r for free.

Example 6. Consider a resource theory of quantum channels where the free resources are classical channels; i.e., quantum channels that are completely dephased in some basis. Define the dimension of a quantum channel to be the dimension of the Hilbert space of the system transmitted by the channel (for channels with different input and output systems, we can choose the smaller one). It is then natural to define a monotone by a cost construction with f the function which returns the dimension of a channel. In this case, the f-cost, or *dimension cost*, of a channel $\mathcal{T}$ is the smallest dimension of channel from which $\mathcal{T}$ can be obtained by composition with free resources. The dimension cost of a channel is upper bounded by its dimension, but in general it can be strictly smaller, e.g., when the channel is completely decohering in some subspace of the Hilbert space.

4.2 f_W-yield and f_W-cost Given a Function f_W Defined on a Subset of Resources

It is often useful to be able to evaluate resources in terms of their cost or yield with respect to a particular set of special resources that one could call a "gold standard".

Let $W \subseteq \mathcal{R}$ denote a subset of resources, and consider a function $f_W\colon W \to \overline{\mathbb{R}}$. It is not hard to see that one can accomodate constructions from the previous section to this case by restricting all optimizations to be within W. Specifically, we can again define functions f_W-max and f_W-min as

$$f_W\text{-max}\colon \mathcal{DC}(\mathcal{R}) \to \overline{\mathbb{R}} \qquad\qquad f_W\text{-min}\colon \mathcal{UC}(\mathcal{R}) \to \overline{\mathbb{R}}$$
$$S \mapsto \sup f(S \cap W) \qquad\qquad S \mapsto \inf f(S \cap W),$$

where $\sup \emptyset := -\infty$ and $\inf \emptyset := \infty$. Therefore, we also get yield and cost monotones on $(\mathcal{R}, \succeq)$ defined by

$$f_W\text{-yield}(r) := f_W\text{-max}(\downarrow r) \qquad\qquad f_W\text{-cost}(r) := f_W\text{-min}(\uparrow r) \tag{16}$$

which can be expressed as

$$f_W\text{-yield}(r) = \sup\{f(s) \mid s \in (\downarrow r) \cap W\} = \sup\{f(s) \mid s \in \mathcal{R}_{\text{free}} \star r,\ s \in W\} \tag{17}$$
$$f_W\text{-cost}(r) = \inf\{f(s) \mid s \in (\uparrow r) \cap W\} = \inf\{f(s) \mid r \in \mathcal{R}_{\text{free}} \star s,\ s \in W\}. \tag{18}$$

The f_W-yield of r is now the largest value of f among the resources within W that can be obtained from r for free, while the f_W-cost of r is the smallest value of f among the resources within W that one can use to obtain r for free.

Example 7. The "currencies" described in [10] are generalized yield and cost monotones, for which W is a chain; i.e., a totally ordered set of resources. A concrete example of this type—from entanglement theory—is the cost of an entangled state measured in the number of e-bits needed to produce it. It is called the single-shot entanglement cost. In that case, W is the set of n-fold tensor products of e-bits for different values of n and f just returns the integer n. Another example—from the classical resource theory of nonuniformity [11]—is the single-shot nonuniformity yield[1] of a statistical state, where W is the set of sharp states, and f is the Shannon nonuniformity.

Example 8. The convex roof extension method, which converts monotones defined for pure states to monotones defined for all mixed states, can be also seen as an instance of the generalized cost construction if we let W be the set of all pure state ensembles and we let f be the convex extension of some monotone for pure-states to pure-state ensembles.

Example 9. In the resource theory of nonclassicality of common-cause boxes [12] the two central monotones—M_{CHSH} and M_{NPR}—are both instances of this type of construction[2]. M_{CHSH} is a generalized yield construction where W is the set of all common-cause boxes with binary inputs and binary outputs and f the CHSH function. M_{NPR} is a generalized cost construction where W is the chain of noisy PR boxes and f is again the CHSH function. Note that M_{CHSH} and M_{NPR} are defined on common-cause boxes with any cardinality of inputs and outputs, even though the CHSH function is not.

Example 10. The constructions of f_W-yield and f_W-cost thus allow one to extend monotones defined for a particular class of resources, such as states, to monotones for other types of resources within the same resource theory, such as channels, measurements, or higher order processes. For instance, in entanglement theory, one can define a monotone on channels from a monotone on states, such as the cost in number of e-bits of implementing a given coherent channel.

Example 11. Generalized channel divergences [13] arise from the generalized cost construction when thinking about the resource theory of pairs of resources. More details on pairs (and other tuples) of resources and in what way they constitute a resource theory can be found in section 5.2.1.

The functions f_W-max and f_W-min are not the only kind of monotones one can define on $(\mathcal{DC}(\mathcal{R}), \supseteq)$ and $(\mathcal{UC}(\mathcal{R}), \subseteq)$ and then pull back to obtain resource monotones on $(\mathcal{R}, \succeq)$. There are other general constructions one can use, which we describe in the full article. With those, one can incorporate even more examples of monotones, such as the *Schmidt number for density matrices* [14], which is a monotone in entanglement theory for bipartite mixed states.

4.3 f_W-yield and f_W-cost Relative to a Downward Closed Set $D \in \mathcal{DC}(\mathcal{R})$

Apart from varying the order-preserving map σ_2 from the Broad Scheme as we have done in section 4.2, one can also vary the order-preserving map σ_1. In particular, $\downarrow$ and $\uparrow$ are not the only order-preserving functions from $\mathcal{R}$ to $\mathcal{DC}(\mathcal{R})$ and $\mathcal{UC}(\mathcal{R})$ respectively.

Definition 12. Consider a set of resources $D \subseteq \mathcal{R}$. We define the **D-image map** $\downarrow_D \colon \mathcal{R} \to \mathcal{DC}(\mathcal{R})$ by

$$\downarrow_D r := \{ s \in \mathcal{R} \mid s \in D \star r \}, \tag{19}$$

for any $r \in \mathcal{R}$, which can be also written as $\downarrow_D r = D \star s$. Similarly, the **D-preimage map**[3] $\uparrow_D \colon \mathcal{R} \to \mathcal{UC}(\mathcal{R})$ is defined by

$$\uparrow_D r := \{ s \in \mathcal{R} \mid r \in D \star s \}. \tag{20}$$

One can show that both $\downarrow_D$ and $\uparrow_D$ are order-preserving, whenever D is a downward closed set of resources. With this notation, we can also see that $\downarrow = \downarrow_{\mathcal{R}_{\text{free}}}$ and $\uparrow = \uparrow_{\mathcal{R}_{\text{free}}}$.

As a consequence of the order-preserving property of $\downarrow_D$ and $\uparrow_D$, we have the following theorem.

[1] Both of the exaples presented here also have their dual counterparts of course. They are called the single-shot entanglement yield and single-shot nonuniformity cost respectively.

[2] Indeed, the monotones M_{CHSH} and M_{NPR} were inspired by the work presented here.

[3] Unlike $\downarrow$ and $\uparrow$, there is not neccesarily a preorder on $\mathcal{R}$ for which the maps $\downarrow_D$ and $\uparrow_D$ are the downward and upward closure operations respectively.

Theorem 13. *Let $(\mathcal{R}, \mathcal{R}_{\text{free}}, \star)$ be a resource theory, let D be a downward closed subset of $\mathcal{R}$ and let $f_W : W \to \overline{\mathbb{R}}$ be a function on $W \subseteq \mathcal{R}$. The f_W-yield relative to the D-image map, $f_W\text{-}\mathsf{yield}_D : \mathcal{R} \to \overline{\mathbb{R}}$, and the f_W-cost relative to the D-preimage map, $f_W\text{-}\mathsf{cost}_D : \mathcal{R} \to \overline{\mathbb{R}}$, defined as*

$$f_W\text{-}\mathsf{yield}_D(r) := f_W\text{-}\mathsf{max}(\downarrow_D r) \qquad f_W\text{-}\mathsf{cost}_D(r) := f_W\text{-}\mathsf{min}(\uparrow_D r) \tag{21}$$

are both resource monotones.

By unpacking the definitions, we can express them as

$$f_W\text{-}\mathsf{yield}_D(r) = \sup\{f(w) \mid w \in D \star r \text{ and } w \in W\} \tag{22}$$
$$f_W\text{-}\mathsf{cost}_D(r) = \inf\{f(w) \mid r \in D \star w \text{ and } w \in W\}. \tag{23}$$

The $f_W\text{-}\mathsf{yield}_D$ of r is the largest value of f among the resources within W that can be obtained from r by composing it with a resource in D. On the other hand, the $f_W\text{-}\mathsf{cost}_D$ of r is the smallest value of f among the resources within W that one can compose with a resource in D and obtain r.

There are two main reasons why one might want use $f_W\text{-}\mathsf{yield}_D$ (or $f_W\text{-}\mathsf{cost}_D$) instead of $f_W\text{-}\mathsf{yield} \equiv f_W\text{-}\mathsf{yield}_{\mathcal{R}_{\text{free}}}$ (or $f_W\text{-}\mathsf{cost} \equiv f_W\text{-}\mathsf{cost}_{\mathcal{R}_{\text{free}}}$). On the one hand, a downward closed set D different from $\mathcal{R}_{\text{free}}$ can be easier to work with either algebraically or numerically when evaluating the function explicitly. This is a common practice in many resource theories in which $\mathcal{R}_{\text{free}}$ is not straightforward to work with. For example, LOCC operations in entanglement theory get replaced by separable operations, noisy operations in nonuniformity theory get replaced by unital operations, and thermal operations in athermality theory get replaced by Gibbs-preserving operations.

On the other hand, $f_W\text{-}\mathsf{yield}_D$ and $f_W\text{-}\mathsf{cost}_D$ can give us new interesting monotones distinct from $f_W\text{-}\mathsf{yield}$ and $f_W\text{-}\mathsf{cost}$, not merely approximations thereof. However, to our knowledge there are no commonly used monotones at present, which are distinctively of this kind. Here we give a simple toy example of how one could use these constructions for $D \neq \mathcal{R}_{\text{free}}$ in practice.

Example 14. Imagine a quantum resource theory in which there are no free states. Such resource theories arise naturally when we consider multi-resource theories [15, 16] for instance. Since a channel can only be converted to a state by feeding a state (or states) into its input (or inputs), there is no way to convert a channel to a state for free in this case. Therefore, evaluating $f_W\text{-}\mathsf{yield}$ for a function f_W, defined on states only, would lead to a trivial monotone for channels. One would not be able to use this construction to extend monotones for states to monotones for channels.

However, one can instead use a downward closed set D that does include some states, in which case $f_W\text{-}\mathsf{yield}_D$ becomes a non-trivial monotone for channels in the resource theory. A choice of D that is guaranteed to be downward closed and include some states is $\mathcal{R}_{\text{free}} \star r$ for a particular state $r \in \mathcal{R}$. $\mathcal{R}_{\text{free}} \star r$ can contain more states than just r of course. In particular, it will contain any other state one could obtain from r for free.

Note that for any set of resources S, the set $S \star \mathcal{R}_{\text{free}}$ is downward closed. It need not be closed under $\star$, in which case it is not a candidate for the set of free resources in a resource theory. Nevertheless, we can use $S \star \mathcal{R}_{\text{free}}$ in generalized yield and cost constructions from theorem 13 by defining the image and preimage maps with respect to it. One way to interpret[4] taking the images and preimages with respect to $S \star \mathcal{R}_{\text{free}}$ is as follows. They quantify yields and costs for an agent who, in addition to having access to the free resources in unlimited supply, also has access to one of the resources from S for a one-time use. Of course, if it *is* the case that $S \star \mathcal{R}_{\text{free}}$ is closed under $\star$, then we can think of $S \star \mathcal{R}_{\text{free}}$ as describing access to both $\mathcal{R}_{\text{free}}$ and S in unlimited supply. In such case we can think of $(\mathcal{R}, S \star \mathcal{R}_{\text{free}}, \star)$ as a resource theory[5].

5 Translating Monotones Between Resource Theories

Let us now change the kind of ordered sets $(\mathcal{A}, \succeq_{\mathcal{A}})$ from the Broad Scheme that we consider when constructing monotones. In section 4, we looked at $\mathcal{A} = \mathcal{DC}(\mathcal{R})$ and $\mathcal{A} = \mathcal{UC}(\mathcal{R})$, and we made use of the

[4] This interpretation is valid if discarding operation is a free resource (or else if S contains 0), but this is true for basically all resource theories that are studied currently.

[5] If $0 \in S$, then $\mathcal{R}_{\text{free}} \subseteq S \star \mathcal{R}_{\text{free}}$ and the resource theory $(\mathcal{R}, \mathcal{R}_{\text{free}}, \star)$ is a subtheory of $(\mathcal{R}, S \star \mathcal{R}_{\text{free}}, \star)$.

fact that the ordering on each corresponds to subset inclusion on $\mathcal{R}$. In the present section, we investigate what can be said about the case when $(\mathcal{A}, \succeq_{\mathcal{A}})$ arises from a resource theory $(\mathcal{Q}, \mathcal{Q}_{\text{free}}, \star_{\mathcal{Q}})$, possibly different from the one we are ultimately interested in obtaining monotones for—$(\mathcal{R}, \mathcal{R}_{\text{free}}, \star_{\mathcal{R}})$.

$(\mathcal{A}, \succeq_{\mathcal{A}})$ could be the set of resources $\mathcal{Q}$ with the ordering $\succeq_{\mathcal{Q}}$ given by the resource conversion relation defined in (5) but relative to the resource theory $(\mathcal{Q}, \mathcal{Q}_{\text{free}}, \star_{\mathcal{Q}})$. However, it could also correspond to the power set $\mathcal{P}(\mathcal{Q})$ with the ordering $\succeq_{\mathcal{Q}}$ defined as in (6) for instance.

5.1 Translating Monotones from a Resource Theory to Itself

First of all, let's look at a particularly simple case of $\mathcal{A} = \mathcal{Q} = \mathcal{R}$. If $\mathcal{Q}_{\text{free}} \supseteq \mathcal{R}_{\text{free}}$, then the identity map

$$\mathsf{Id} \colon (\mathcal{R}, \succeq) \to (\mathcal{Q}, \succeq_{\mathcal{Q}}) \tag{24}$$

is order-preserving, since $\mathcal{Q}_{\text{free}}$ allows for more conversions than $\mathcal{R}_{\text{free}}$. Therefore, any monotone with respect to $\mathcal{Q}_{\text{free}}$ is also a monotone with respect to $\mathcal{R}_{\text{free}}$.

Example 15. In entanglement theory, one can take $\mathcal{R}_{\text{free}}$ to be the set of LOCC operations and $\mathcal{Q}_{\text{free}}$ to be the set of separable operations [17], which is a strict superset of LOCC. Any monotone relative to the separable operations is then also a monotone relative to LOCC.

Example 16. In the resource theory of athermality, one can take $\mathcal{R}_{\text{free}}$ to be the set of thermal operations and $\mathcal{Q}_{\text{free}}$ to be the set of Gibbs-preserving operations [18], which is generally a strict superset of the thermal operations. Any monotone relative to Gibbs-preserving operations is then also a monotone relative to thermal operations.

5.1.1 Monotones for Sets of Resources

Other interesting cases arise even when we consider $\mathcal{R}_{\text{free}} = \mathcal{Q}_{\text{free}}$, so that there is no distinction between the resource theory $\mathcal{R}$ that is the target for our monotone constructions and the resource theory $\mathcal{Q}$ that is the source of monotones. Instead of $\mathcal{A} = \mathcal{R}$, however, we choose $\mathcal{A} = \mathcal{P}(\mathcal{R})$ and consider two orderings $\succeq_{\mathcal{A}}$.

Before we describe these two possibilities for $\succeq_{\mathcal{A}}$, we need to extend the free image and free preimage maps introduced in section 3 to sets of resources. We define $\downarrow \colon \mathcal{P}(\mathcal{R}) \to \mathcal{DC}(\mathcal{R})$ and $\uparrow \colon \mathcal{P}(\mathcal{R}) \to \mathcal{UC}(\mathcal{R})$ to act as follows. For any set of resources S, its free image, denoted $\downarrow S$, is the union of images of elements of S under the free image map for individual resources. Likewise, its free preimage, denoted $\uparrow S$, is the union of images of elements of S under the free preimage map for individual resources.

The first choice of $\succeq_{\mathcal{A}}$ we consider is $\succeq_{\text{enh}}$, defined by

$$S \succeq_{\text{enh}} T \quad \iff \quad \exists \mathsf{enh} \colon T \to S \text{ such that for all } r \in S \text{ we have } \mathsf{enh}(r) \succeq r, \tag{25}$$

where enh is a function that we term an enhancement function, because it maps resources to ones of higher (or equal) value according to the resource ordering. Our second choice of $\succeq_{\mathcal{A}}$ is $\succeq_{\text{deg}}$, defined by

$$S \succeq_{\text{deg}} T \quad \iff \quad \exists \mathsf{deg} \colon S \to T \text{ such that for all } r \in S \text{ we have } r \succeq \mathsf{deg}(r), \tag{26}$$

where deg is a function that we term a degradation function, because it maps resources to ones of lower (or equal) value according to the resource ordering. An equivalent presentation can be given in terms of the maps $\downarrow$ and $\uparrow$. In particular, $S \succeq_{\text{enh}} T \iff \downarrow S \supseteq \downarrow T$ and $S \succeq_{\text{deg}} T \iff \uparrow S \subseteq \uparrow T$. This also means that

$$\downarrow \colon (\mathcal{P}(\mathcal{R}), \succeq_{\text{enh}}) \to (\mathcal{DC}(\mathcal{R}), \supseteq) \qquad\qquad \uparrow \colon (\mathcal{P}(\mathcal{R}), \succeq_{\text{deg}}) \to (\mathcal{UC}(\mathcal{R}), \subseteq) \tag{27}$$

are both order-preserving and define monotones $f\text{-}\mathsf{max} \circ \downarrow$ and $f\text{-}\mathsf{min} \circ \uparrow$ on $\mathcal{P}(\mathcal{R})$ for any function $f \colon \mathcal{R} \to \overline{\mathbb{R}}$. If f happens to be a monotone in its own right, we find that[6] $f\text{-}\mathsf{max} \circ \downarrow = f\text{-}\mathsf{max}$ and $f\text{-}\mathsf{min} \circ \uparrow = f\text{-}\mathsf{min}$, in which case we can take

$$f\text{-}\mathsf{max} \colon (\mathcal{P}(\mathcal{R}), \succeq_{\text{enh}}) \to (\overline{\mathbb{R}}, \geq) \qquad\qquad f\text{-}\mathsf{min} \colon (\mathcal{P}(\mathcal{R}), \succeq_{\text{deg}}) \to (\overline{\mathbb{R}}, \geq) \tag{28}$$

to play the role of the monotone σ_2 in the Broad Scheme for the two distinct choices of $\succeq_{\mathcal{A}}$.

[6]The maps $f\text{-}\mathsf{max}$ and $f\text{-}\mathsf{min}$ on the right hand sides of these equalities are the obvious extensions of the maps defined in section 4.1 to $\mathcal{P}(\mathcal{R})$. That is, $f\text{-}\mathsf{max}(S) = \sup f(S)$ and $f\text{-}\mathsf{min}(S) = \inf f(S)$

5.1.2 Order-Preserving Maps from $\mathcal{R}$ to $\mathcal{P}(\mathcal{R})$

We need to describe the order-preserving functions $\mathcal{R} \to \mathcal{P}(\mathcal{R})$ one could use as σ_1. Since $\succeq_{\text{enh}}$ is different from $\succeq_{\text{deg}}$, we have to find distinct order-preserving maps $(\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{R}), \succeq_{\text{enh}})$ and $(\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{R}), \succeq_{\text{deg}})$ in order to be able to use both f-max and f-min in the Broad Scheme.

Example 17. Given a resource $c \in \mathcal{R}$, consider the map $\mathsf{Sup}_c \colon (\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{R}), \succeq_{\text{enh}})$ termed the **supplementation by c**[7] that is defined for any resource r by

$$\mathsf{Sup}_c(r) \coloneqq c \star r. \tag{29}$$

This map is always order-preserving, since the implication $r \succeq s \implies c \star r \succeq c \star s$ follows from the definition of a resource theory and the resource ordering for any $r, s, c \in \mathcal{R}$. Furthermore, as we show in the full paper, $(\mathcal{P}(\mathcal{R}), \succeq_{\text{enh}})$ is equal to $(\mathcal{P}(\mathcal{R}), \succeq)$ with $\succeq$ defined as in (6). The same works when c is not just a single resource, but a set of resources.

Example 18. Consider the map $\mathsf{Copy}_2 \colon (\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{R}), \succeq_{\text{enh}})$ defined by

$$\mathsf{Copy}_2(r) \coloneqq r \star r. \tag{30}$$

Given that $s \in \mathcal{R}_{\text{free}} \star r$ implies $s \star s \in (\mathcal{R}_{\text{free}} \star r) \star (\mathcal{R}_{\text{free}} \star r) = \mathcal{R}_{\text{free}} \star (r \star r)$, it follows that $s \succeq r$ implies $s \star s \succeq r \star r$, and consequently Copy_2 is an order-preserving map. The same works for the map $\mathsf{Copy}_n \colon \mathcal{R} \to \mathcal{P}(\mathcal{R})$ defined by $\mathsf{Copy}_n(r) = r^{\star n}$ (i.e., the n-fold $\star$-product).

Both Sup_c and Copy_n are thus examples of order-preserving maps from $(\mathcal{R}, \succeq)$ to $(\mathcal{P}(\mathcal{R}), \succeq_{\text{enh}})$ and can be used to obtain monotones f-$\mathsf{max} \circ \mathsf{Sup}_c$ and f-$\mathsf{max} \circ \mathsf{Copy}_n$ for any monotone f, any resource c and any integer n. These monotone constructions differ from the ones we have seen in section 4 in that the optimization is generally restricted to range over a much smaller set of resources.

General sufficient conditions for a function $(\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{R}), \succeq_{\text{enh}})$ to be order-preserving are provided by the following lemma, whose proof is given in the full article. Note that both Sup_c and Copy_n satisfy conditions (47) and (48) basically by definition.

Lemma 19. *Let $(\mathcal{R}, \mathcal{R}_{\text{free}}, \star)$ be a resource theory and let $F \colon \mathcal{R} \to \mathcal{P}(\mathcal{R})$ be a function with an extension $F \colon \mathcal{P}(\mathcal{R}) \to \mathcal{P}(\mathcal{R})$ obtained from the original F by requiring that it commutes with unions*[8]*. If we have*

$$F(r \star s) \subseteq F(r) \star F(s) \quad \forall r, s \in \mathcal{R}, \text{ and} \tag{31}$$

$$F(\mathcal{R}_{\text{free}}) \subseteq \mathcal{R}_{\text{free}} \star F(0), \tag{32}$$

then $F \colon (\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{R}), \succeq_{\text{enh}})$ is order-preserving.

Non-trivial order-preserving maps from $(\mathcal{R}, \succeq)$ to $(\mathcal{P}(\mathcal{R}), \succeq_{\text{deg}})$ are harder to come by. Nevertheless, as example 20 shows, they do arise. Furthermore, when $\mathcal{Q}$ is distinct from $\mathcal{R}$, they become very important as we will see in section 5.3 and especially in section 5.2.2.

Example 20. Notice that the D-image map $\downarrow_D$ and the D-preimage map $\uparrow_D$ defined in section 4.3, when thought of as maps $(\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{R}), \succeq_{\text{enh}})$ and $(\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{R}), \succeq_{\text{deg}})$ respectively, are order-preserving whenever $D \in \mathcal{DC}(\mathcal{R})$. We can therefore arrive at theorem 13 from this perspective as well.

5.2 Translating Monotones from Information Theory

Many resource theories of interest have an information-theoretic flavour or are explicitly about informational resources. It is no surprise then, that in these resource theories measures of information often crop up as monotones or as building blocks for resource monotones. As we will see below, to any resource theory $(\mathcal{R}, \mathcal{R}_{\text{free}}, \star)$, it is possible to associate a resource theory of information encodings. This association can then be used to understand such results in greater generality.

[7] We can think of the action of Sup_c as combination of resources with a catalyst c.

[8] The extension $F \colon \mathcal{P}(\mathcal{R}) \to \mathcal{P}(\mathcal{R})$ maps a set S to the union of images of elements of S under $F \colon \mathcal{R} \to \mathcal{P}(\mathcal{R})$.

Example 21 (Monotones from contractions)**.** The key example of a monotone construction that we aim to understand and generalize here is one based on contractions. A contraction in a quantum resource theory of states, for example, is a function f of pairs of quantum states that satisfies the data processing inequality

$$f(\rho,\sigma) \geq f(\Phi(\rho),\Phi(\sigma)) \tag{33}$$

for all states ρ and σ (of the same system) and all CPTP maps Φ. Given such a contraction, it is well-known that one can obtain a monotone by minimizing over all free states in one of its arguments. That is, the function $M\colon \mathcal{R} \to \overline{\mathbb{R}}$ given by

$$M(\rho) = \inf\big\{ f(\rho,\sigma) \;\big|\; \sigma \in \mathcal{R}_{\text{free}} \big\} \tag{34}$$

is a resource monotone. Various monotones based on distance measures such as the trace distance, relative Rényi entropies [19], and many others arise in this way. An extensive overview of these kinds of monotones can be found in [6].

5.2.1 Resource Theory of Encodings of Classical Information

In order to understand the construction from example 21 via the Broad Scheme and generalize it, we study a resource theory of tuples of resources, in which contractions become its monotones.

Definition 22. Let k be a natural number and consider a resource theory $(\mathcal{R}, \mathcal{R}_{\text{free}}, \star)$ with $r \star s$ being a single resource for all $r, s \in \mathcal{R}$. The **resource theory of encodings of a classical k-ary hypothesis in** $\mathcal{R}$ is a resource theory $(\mathcal{R}^k, \mathcal{R}^k_{\text{cons}}, \star_k)$, where

1. $\mathcal{R}^k = \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R}$ is the set of all k-tuples of resources from $\mathcal{R}$,
2. $\mathcal{R}^k_{\text{cons}} \subseteq \mathcal{R}^k$ is the set of all constant k-tuples, and
3. the composition of k-tuples is given by

$$(r_1, r_2, \ldots, r_k) \star_k (s_1, s_2, \ldots, s_k) = (r_1 \star s_1, r_2 \star s_2, \ldots, r_k \star s_k) \tag{35}$$

   where we think of $\star$ as a map $\mathcal{R} \times \mathcal{R} \to \mathcal{R}$.

One can construct a similar resource theory even under much weaker assumptions than the fact that any two resources can be combined in exactly one way. However, the construction becomes more complicated and therefore we forego considering it here in order to avoid obscuring the main ideas.

As the name of the resource theory $(\mathcal{R}^k, \mathcal{R}^k_{\text{cons}}, \star_k)$ suggests, we can think of the k-tuples as encodings of a classical hypothesis. Namely, if H_k is a classical hypothesis with cardinality k, a k-tuple of resources is a map $H_k \to \mathcal{R}$. The "free" k-tuples are the constant ones because they can be constructed with no information about the value of H_k. That is, if the k-tuple in question is from $\mathcal{R}^k_{\text{cons}}$, so that every value of H_k is associated to the *same* resource, then learning the identity of the resource teaches one nothing about the value of H_k. Note that this restriction doesn't distinguish between valuable and free resources in the original theory $(\mathcal{R}, \mathcal{R}_{\text{free}}, \star)$. The resource theory $(\mathcal{R}^k, \mathcal{R}^k_{\text{cons}}, \star_k)$ is purely about the information content of the encodings.

5.2.2 Monotones from Contractions in General

Because of the optimization in (34), which occurs at the level of *sets* of 2-tuples, we have to use $\mathcal{A} = \mathcal{P}(\mathcal{R}^2)$ rather than $\mathcal{A} = \mathcal{R}^2$ if we want to make sense of the monotone M from example 21 via the Broad Scheme.

As we have seen in section 5.1.1, given any monotone $f\colon (\mathcal{R}^2, \succeq) \to (\overline{\mathbb{R}}, \geq)$, we have monotones $f\text{-}\mathsf{max}\colon (\mathcal{P}(\mathcal{R}^2), \succeq_{\text{enh}}) \to (\overline{\mathbb{R}}, \geq)$ and $f\text{-}\mathsf{min}\colon (\mathcal{P}(\mathcal{R}^2), \succeq_{\text{deg}}) \to (\overline{\mathbb{R}}, \geq)$. These will constitute our choice of σ_2 from the Broad Scheme once again.

In order to obtain monotones on $\mathcal{R}$ from f-max and f-min, we need order-preserving maps from $(\mathcal{R}, \succeq)$ to $(\mathcal{P}(\mathcal{R}^2), \succeq_{\text{enh}})$ and $(\mathcal{P}(\mathcal{R}^2), \succeq_{\text{deg}})$ respectively. For both orderings on $\mathcal{P}(\mathcal{R}^2)$, the map $\mathcal{E}\colon \mathcal{R} \to \mathcal{P}(\mathcal{R}^2)$ defined by

$$\mathcal{E}(r) := \{(r,s) \;|\; s \in \mathcal{R}\} \tag{36}$$

is order-preserving, which we prove in the full article. Furthermore, for any upward closed set $\mathcal{R}_{\mathrm{uc}} \in \mathcal{UC}(\mathcal{R})$, the map $\mathcal{E}_{\mathcal{R}_{\mathrm{uc}}} \colon (\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{R}^2), \succeq_{\mathrm{enh}})$ defined by

$$\mathcal{E}_{\mathcal{R}_{\mathrm{uc}}}(r) \coloneqq \{(r, s) \mid s \in \mathcal{R}_{\mathrm{uc}}\} \tag{37}$$

is order-preserving. Likewise, for any downward closed set $\mathcal{R}_{\mathrm{dc}}$, the map $\mathcal{E}_{\mathcal{R}_{\mathrm{dc}}} \colon (\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{R}^2), \succeq_{\mathrm{deg}})$ defined by

$$\mathcal{E}_{\mathcal{R}_{\mathrm{dc}}}(r) \coloneqq \{(r, s) \mid s \in \mathcal{R}_{\mathrm{dc}}\} \tag{38}$$

is order-preserving.

The upshot of this discussion is the following theorem, a special case of a significantly more general result to be found in the full article.

Theorem 23. *Let $f \colon \mathcal{R}^2 \to \overline{\mathbb{R}}$ be a monotone in $(\mathcal{R}^2, \mathcal{R}^2_{\mathrm{free}}, \star_2)$. For any upward closed subset $\mathcal{R}_{\mathrm{uc}}$ of $\mathcal{R}$ and any downward closed subset $\mathcal{R}_{\mathrm{dc}}$ of $\mathcal{R}$, the two functions $f\text{-}\mathsf{max} \circ \mathcal{E}_{\mathcal{R}_{\mathrm{uc}}}$ and $f\text{-}\mathsf{min} \circ \mathcal{E}_{\mathcal{R}_{\mathrm{dc}}}$ given by*

$$f\text{-}\mathsf{max} \circ \mathcal{E}_{\mathcal{R}_{\mathrm{uc}}}(r) = \sup\{f(r, s) \mid s \in \mathcal{R}_{\mathrm{uc}}\} \tag{39}$$

$$f\text{-}\mathsf{min} \circ \mathcal{E}_{\mathcal{R}_{\mathrm{dc}}}(r) = \inf\{f(r, s) \mid s \in \mathcal{R}_{\mathrm{dc}}\} \tag{40}$$

are monotones on $(\mathcal{R}, \mathcal{R}_{\mathrm{free}}, \star)$.

Consequently, we obtain M from example 21 as $f\text{-}\mathsf{min} \circ \mathcal{E}_{\mathcal{R}_{\mathrm{dc}}}$ with $\mathcal{R}_{\mathrm{dc}}$ being the downward closed set of all the free resources—$\mathcal{R}_{\mathrm{free}}$.

5.2.3 Resource Weight and Robustness as Monotones Obtained from a Contraction

As an example of how the constructions from theorem 23 appear in a more concrete setting, we examine arguably two of the most ubiquitous monotones—resource weight and robustness—within the context of resource theories with a linear structure.

Let's consider a resource theory $(\mathcal{R}, \mathcal{R}_{\mathrm{free}}, \star)$ with a linear structure on $\mathcal{R}$ that is preserved by $\star$. The elements of $\mathcal{R}$ can thus be represented as vectors, and convex combination is preserved by composition of resources. Furthermore, just like in the previous section, we assume that $r \star s$ is a single resource for all $r, s \in \mathcal{R}$, so that $\star$ is a bilinear map $\mathcal{R} \times \mathcal{R} \to \mathcal{R}$. A more general scenario corresponding to a broader idea of linear resource theories is treated in the full paper.

We can construct the resource theory of 3-tuples $(\mathcal{R}^3, \mathcal{R}^3_{\mathrm{cons}}, \star_3)$ as described in section 5.2.1 and define the following function called the **convex alignment**.

Definition 24. Let $\mathsf{cva} \colon \mathcal{R}^3 \to \overline{\mathbb{R}}$ be the function defined by

$$\mathsf{cva}(r, s, t) \coloneqq \begin{cases} \lambda & \text{if } r = \lambda s + (1 - \lambda) t \text{ for } \lambda \in [0, 1]. \\ 1 & \text{otherwise.} \end{cases} \tag{41}$$

Lemma 25. *Let $(\mathcal{R}, \mathcal{R}_{\mathrm{free}}, \star)$ be a resource theory with a linear structure as described at the start of this section. The convex alignment, cva, is a monotone in the resource theory $(\mathcal{R}^3, \mathcal{R}^3_{\mathrm{cons}}, \star_3)$.*

Lemma 25 holds basically by definition. Since we assumed that resource combination $\star$ preserves convex mixtures, the value of cva cannot increase under composition of (r, s, t) with any element of $\mathcal{R}^3_{\mathrm{cons}}$ under $\star_3$.

Now we can use theorem 23, or rather a generalization thereof for 3-tuples, to get monotones for $(\mathcal{R}, \mathcal{R}_{\mathrm{free}}, \star)$ by optimizing cva in various ways. Let us first focus on a construction wherein one performs minimization over downward closed sets on *two* elements of the 3-tuple, a generalization of the construction $f\text{-}\mathsf{min} \circ \mathcal{E}_{\mathcal{R}_{\mathrm{dc}}}$ in theorem 23. There are many downward closed sets one could use for each element of the 3-tuples, but here we will focus on the two most obvious choices—$\mathcal{R}_{\mathrm{free}}$ and $\mathcal{R}$. Even with this restriction, one obtains 12 constructions analogous to $f\text{-}\mathsf{min} \circ \mathcal{E}_{\mathcal{R}_{\mathrm{dc}}}$, 8 of which are constant and therefore uninteresting. The other 4 are the following.

1. The **resource weight** $M_{\mathrm{w}} \colon \mathcal{R} \to \overline{\mathbb{R}}$ is defined as

$$\begin{aligned} M_{\mathrm{w}}(r) &\coloneqq \inf\bigl\{\mathsf{cva}(r, s, t) \bigm| s \in \mathcal{R},\ t \in \mathcal{R}_{\mathrm{free}}\bigr\} \\ &= \inf\bigl\{\lambda \bigm| r \in \lambda \mathcal{R} + (1 - \lambda)\mathcal{R}_{\mathrm{free}}\bigr\}. \end{aligned} \tag{42}$$

2. The **resource robustness** $M_{\text{rob}} \colon \mathcal{R} \to \overline{\mathbb{R}}$ is defined as

$$\begin{aligned} M_{\text{rob}}(t) &\coloneqq \inf\big\{ \mathsf{cva}(r,s,t) \bigm| r \in \mathcal{R}_{\text{free}},\ s \in \mathcal{R} \big\} \\ &= \inf\big\{ \lambda \bigm| \lambda\mathcal{R} + (1-\lambda)t \in \mathcal{R}_{\text{free}} \big\}. \end{aligned} \tag{43}$$

3. The **free robustness** $M_{\text{f. rob}} \colon \mathcal{R} \to \overline{\mathbb{R}}$ is defined as

$$\begin{aligned} M_{\text{f. rob}}(t) &\coloneqq \inf\big\{ \mathsf{cva}(r,s,t) \bigm| r \in \mathcal{R}_{\text{free}},\ s \in \mathcal{R}_{\text{free}} \big\} \\ &= \inf\big\{ \lambda \bigm| \lambda\mathcal{R}_{\text{free}} + (1-\lambda)t \in \mathcal{R}_{\text{free}} \big\}. \end{aligned} \tag{44}$$

4. The **resource non-convexity** $M_{\text{nc}} \colon \mathcal{R} \to \overline{\mathbb{R}}$ is defined as

$$\begin{aligned} M_{\text{nc}}(r) &\coloneqq \inf\big\{ \mathsf{cva}(r,s,t) \bigm| s \in \mathcal{R}_{\text{free}},\ t \in \mathcal{R}_{\text{free}} \big\} \\ &= \inf\big\{ \lambda \bigm| r \in \lambda\mathcal{R}_{\text{free}} + (1-\lambda)\mathcal{R}_{\text{free}} \big\}. \end{aligned} \tag{45}$$

As a consequence of the generalization of theorem 23 found in the full article and lemma 25, all four functions above are monotones. However, being able to prove the monotonicity of these four functions is not where the value of theorem 23 lies. What it provides is an understanding of the assumptions required in order for these to be monotones. Furthermore, it gives us a unified picture, within which we can adjust various elements of the monotone construction according to the question we are interested in. In this case, there are many more monotones one can obtain from cva in this way, since $\mathcal{R}$ or $\mathcal{R}_{\text{free}}$ in the optimization can be replaced by any other downward closed set.

The above discussion gives an example of how one could use one half of theorem 23. What about the other half? In order to apply the construction arising from f-max, we need an upward closed set $U \in \mathcal{DC}(\mathcal{R}^3)$. However, $\mathcal{R}_{\text{free}}$ is not upward closed and using $U = \mathcal{R} \times \mathcal{R} \times \mathcal{R}$ gives a trivial—i.e., constant—monotone. Therefore, we cannot use any of the sets we used when we constructed the resource weight, robustness, and related monotones from cva. In concrete resource theories, there are nevertheless many upward closed sets one could use.

For example, if $\mathcal{R}$ is a convex set, it can often be the case that the boundary of $\mathcal{R}$, denoted by $\mathcal{B}$, is an upward closed set. This occurs whenever combining a resource in the interior of $\mathcal{R}$ with a free resource cannot produce a resource at the boundary of $\mathcal{R}$. In such a case, we get the following monotone.

1. The **resource purity** $M_{\text{pur}} \colon \mathcal{R} \to \overline{\mathbb{R}}$ is defined as

$$\begin{aligned} M_{\text{pur}}(r) &= \sup\big\{ \mathsf{cva}(r,s,t) \bigm| s \in \mathcal{B},\ t \in \mathcal{B} \big\} \\ &= \sup\big\{ \lambda \bigm| r = \lambda s + (1-\lambda)t \text{ and } s,t \in \mathcal{B} \big\}. \end{aligned} \tag{46}$$

5.3 General Ways of Translating Monotones from a Resource Theory

In section 5.1 we investigated how one can translate monotones from a resource theory $(\mathcal{Q}, \mathcal{Q}_{\text{free}}, \star_{\mathcal{Q}})$ to a resource theory $(\mathcal{R}, \mathcal{R}_{\text{free}}, \star_{\mathcal{R}})$ when the two are in fact the same or, at the very least, $\mathcal{Q} = \mathcal{R}$. Then, in section 5.2, we looked at the choice of $(\mathcal{Q}, \mathcal{Q}_{\text{free}}, \star_{\mathcal{Q}})$ being a kind of information theory—the resource theory of encodings of a classical k-ary hypothesis in $\mathcal{R}$. Here, we would like explore what can be said in general. Can the methods introduced before be extended to the case of arbitrary $\mathcal{Q}$?

The choices of $(\mathcal{A}, \succeq_{\mathcal{A}})$ we consider are $(\mathcal{P}(\mathcal{Q}), \succeq_{\text{enh}})$ and $(\mathcal{P}(\mathcal{Q}), \succeq_{\text{deg}})$. For any monotone f on $\mathcal{Q}$, we again have corresponding monotones f-$\mathsf{max} \colon (\mathcal{P}(\mathcal{Q}), \succeq_{\text{enh}}) \to (\overline{\mathbb{R}}, \geq)$ and f-$\mathsf{min} \colon (\mathcal{P}(\mathcal{Q}), \succeq_{\text{deg}}) \to (\overline{\mathbb{R}}, \geq)$.

In order to find out which maps can be used as σ_1 in the former case; i.e., as an order-preserving map $(\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{Q}), \succeq_{\text{enh}})$, we can use the following conditions that generalize lemma 19.

Lemma 26. *Let $(\mathcal{R}, \mathcal{R}_{\text{free}}, \star)$ and $(\mathcal{Q}, \mathcal{Q}_{\text{free}}, \star_{\mathcal{Q}})$ be resource theories and let $F \colon \mathcal{R} \to \mathcal{P}(\mathcal{Q})$ be a function, with the natural extension to $\mathcal{P}(\mathcal{R})$ as in lemma 19. If we have*

$$F(r \star s) \subseteq F(r) \star_{\mathcal{Q}} F(s) \quad \forall r,s \in \mathcal{R}, \text{ and} \tag{47}$$

$$F(\mathcal{R}_{\text{free}}) \subseteq \mathcal{Q}_{\text{free}} \star_{\mathcal{Q}} F(0), \tag{48}$$

then $F \colon (\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{Q}), \succeq_{\text{enh}})$ is order-preserving.

The function $\mathcal{E}$ introduced in section 5.2.2 is an example of an order-preserving map $(\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{Q}), \succeq_{\text{deg}})$. Notice that we can view it as mapping r to its preimage under the projection $(r, s) \mapsto r$. The following lemma provides sufficient conditions for such functions to be order-preserving in general.

Lemma 27. *Let $(\mathcal{R}, \mathcal{R}_{\text{free}}, \star)$ and $(\mathcal{Q}, \mathcal{Q}_{\text{free}}, \star_{\mathcal{Q}})$ be resource theories and let $F\colon \mathcal{R} \to \mathcal{P}(\mathcal{Q})$ be a function. If there exists a map $G\colon \mathcal{Q} \to \mathcal{R}$ satisfying*

$$F(r) = G^{-1}(r), \tag{49}$$

$$G(p \star_{\mathcal{Q}} q) \supseteq G(p) \star G(q) \quad \forall p, q \in \mathcal{Q}, \text{ and} \tag{50}$$

$$G(\mathcal{Q}_{\text{free}}) \supseteq \mathcal{R}_{\text{free}}, \tag{51}$$

then $F\colon (\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{Q}), \succeq_{\text{deg}})$ is order-preserving.

One can check that $\mathcal{E}$ indeed satisfies these conditions if G is the aforementioned projection $(r, s) \mapsto r$. Furthermore, $\mathcal{E}$ also satisfies conditions (47, 48) in lemma 26, whence it is order-preserving also as a map $(\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{Q}), \succeq_{\text{enh}})$.

Lemma 27 cannot be used, however, to show that the maps $\mathcal{E}_{\mathcal{R}_{\text{uc}}}$ introduced in section 5.2.2 are order-preserving. We need another lemma, with which we have all the ingredients for a method of translating monotones between arbitrary resource theories. A method that subsumes all of the examples we have seen in section 5 until now.

Lemma 28. *Let $(\mathcal{R}, \mathcal{R}_{\text{free}}, \star)$ and $(\mathcal{Q}, \mathcal{Q}_{\text{free}}, \star_{\mathcal{Q}})$ be resource theories and let $W_{\text{uc}} \in \mathcal{UC}(\mathcal{Q})$ and $W_{\text{dc}} \in \mathcal{DC}(\mathcal{Q})$ be upward and downward closed subsets of $\mathcal{Q}$, respectively.*

1. *If $F\colon (\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{Q}), \succeq_{\text{enh}})$ is an order-preserving map, then the map $F_{W_{\text{uc}}}$ defined by*

$$F_{W_{\text{uc}}}(r) \coloneqq F(r) \cap W_{\text{uc}} \tag{52}$$

   is also order-preserving as a map $(\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{Q}), \succeq_{\text{enh}})$.

2. *If $F\colon (\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{Q}), \succeq_{\text{deg}})$ is an order-preserving map, then the map $F_{W_{\text{dc}}}$ defined by*

$$F_{W_{\text{dc}}}(r) \coloneqq F(r) \cap W_{\text{dc}} \tag{53}$$

   is also order-preserving as a map $(\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{Q}), \succeq_{\text{deg}})$.

Given the choice of $(\mathcal{Q}, \mathcal{Q}_{\text{free}}, \star_{\mathcal{Q}}) = (\mathcal{R}^2, \mathcal{R}^2_{\text{free}}, \star_2)$, $F = \mathcal{E}$, $W_{\text{uc}} = \mathcal{R} \times \mathcal{R}_{\text{uc}}$, and $W_{\text{dc}} = \mathcal{R} \times \mathcal{R}_{\text{dc}}$ in lemma 28, we thus recover the fact that $\mathcal{E}_{\mathcal{R}_{\text{uc}}}\colon (\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{R}^2), \succeq_{\text{enh}})$ and $\mathcal{E}_{\mathcal{R}_{\text{dc}}}\colon (\mathcal{R}, \succeq) \to (\mathcal{P}(\mathcal{R}^2), \succeq_{\text{deg}})$ are both order-preserving, which was used in theorem 23 without proof.

The proofs of all of these results can be found in the full article.

6 Conclusions

To summarize, in section 2, we introduced a somewhat minimal framework for describing resource theories, within which we investigate various ways of constructing monotones. The first of these is given by the generalized resource yield and generalized resource cost constructions, which are oulined in section 4. There, we give an extensive but by no means exhaustive list of examples of these constructions appearing in the literature.

Section 5 then deals with monotones which can be seen as arising from another monotone by virtue of a translation via an order-preserving map. In section 5.2.1, we introduce a resource theory of tuples of resources and use it to obtain a monotone construction in terms of contractions. An example of how the theorem on contraction-based monotones can be used is provided in section 5.2.3. There, we unify resource weight and resource robustness as arising from a single contraction for 3-tuples of resources in resource theories with a linear structure. Moreover, by varying the parameters in the general construction, we show how one can obtain other related monotones. General methods to translate monotones between resource theories are then presented in section 5.3, where we also see how all the previous ones arise as special cases.

The two main themes of the paper, generalizing yield and cost constructions and generalizing the translation of monotones, are intricately linked. We investigate these connections further in the full article. For example, we prove that $(\mathcal{DC}(\mathcal{R}), \supseteq)$ is isomorphic (as an ordered set) to $(\mathcal{P}(\mathcal{R})/\!\sim, \succeq_{\text{enh}})$, where $\mathcal{P}(\mathcal{R})/\!\sim$ denotes the equivalence classes of $\mathcal{P}(\mathcal{R})$ with respect to its ordering $\succeq_{\text{enh}}$. Similarly, we also have $(\mathcal{UC}(\mathcal{R}), \subseteq) \cong (\mathcal{P}(\mathcal{R})/\!\sim, \succeq_{\text{deg}})$. Therefore, the choices of $(\mathcal{A}, \succeq_{\mathcal{A}})$ are not as different as they might appear at first sight. What distinguishes them is that the function f, a starting point in methods of both section 4 and section 5, has different properties in the two cases. Section 5 is targeted to functions f which are monotones themselves, while section 4 works for functions f which have a restricted domain or are not even monotones. One could think that the generalized yield and cost constructions from section 4 are therefore superior. However, their disadvantage is the optimization inherent in the construction. Moreover, even though f has to be a monotone in section 5, it can be a seemingly insipid one like the convex alignement cva and we still get interesting monotones from it, as section 5.2.3 demonstrates.

Our work advances the studies of general structures appearing in theories of resources and has potential applications to any area where resource-theoretic thinking is of some use. That means the study of information theory, both quantum and classical, but also of thermodynamics, of renormalization, and of various parts of physics where these are used. There might also be some applications in fields which are more distant, such as machine learning and economics.

There are many possible future directions for this work. Some have already been initiated, the first and third in the following list in particular, even though they made it into this extended abstract only in limited extent.

1. It would be useful to understand the structure of the set of all monotonic functions themselves in order to compare them in terms of how good they are in capturing the resource ordering or by other criteria. The constructions presented here are very general and widely applicable. However, using them in practice as a method for generating monotones involves several choices. For example , a priori it is not clear which choices of D, f (and its domain W) in theorem 13 are the best ones. These are the kind of questions we hope to be able to answer with a better understanding of the relations between monotones.

2. One might also wonder how the mathematical structures presented here relate to other mathematical structures used in physics, mathematics, and computer science. The study of their relation to other mathematical frameworks for resource theories is the topic of a forthcoming paper [9], but other connections are to be developed.

3. Another possible future direction is to try to devise general techniques for constructing monotones by considering resource theories that have more structure than we have presumed here. One way to do so would be by strenghtening assumptions of the general results in order to arrive at stronger conclusions. There are many possibilities in this direction, one of which is to assume a linear or convex structure of resources, as we did in section 5.2.3 here. It is clear that these results will then be connected to ideas from convex geometry and convex optimization [20].

4. Last, but not least, one would hope to be able to not only unify existing concrete results about resource theories as we have done here, but also to find genuinely new results with the help of the conceptual clarity arising from the abstract point of view.

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