Classical theories with entanglement

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We present the framework of Operational Probabilistic Theories (OPTs), where the probabilistic structure hinges upon braided strict monoidal categories. We then focus on classical theories, namely theories where the classes of states are simplexes. We show that, in general, a classical theory may admit entangled states, i.e. states which cannot be prepared with local operations and classical communication. This feature is found to be equivalent to the failure of local discriminability or, equivalently in the classical case, of atomicity of state-composition. We then construct an explicit example—complete with the set of operations—of such a theory. The OPT is causal and allows for both reversible permutability and perfect discriminability of pure states. Interestingly, this theory provides a "counterexample" to the notion of purity defined in the context of process theories (via the so-called *leaks*), meaning that the definition fails to apply to the present case. Entanglement, on the other hand, is found to be a ubiquitous feature for any simplectic theory which is not standard classical theory. Finally, the OPT constructed is well-suited for cryptographic and computational protocols.

1 Probabilistic Theories: Categorical and Operational

The framework of Generalized Probabilistic Theories (GPTs) has the preeminent purpose of providing a suitable toolbox to investigate the probabilistic structure of physical theories. In Ref. [6], the framework has been defined in terms of systems, operations, and correlations. In particular, the main concern of GPTs is the study of the state-space for the systems of a theory, namely of convex sets whereon probability distributions for outcomes are defined. This line of research tracks back to the literature investigating correlations which are stronger than those admitted by both Classical and Quantum Theory (CT and QT), such as those arisen in the popular PR-boxworld and its implementation [37, 36].

GPT has thus represented a broad umbrella-term subsuming various approaches to (physical) theories, with a particular focus on the correlations that can possibly show up. Some guiding principles has been formulated in order to bound the admissible probability spaces for a well-behaved physical theory, such as *causality* and *no-signalling* [18, 22, 21, 17], *incompatibility of observables* [35], (various forms of) *non-locality* [37, 27], *contextuality* [46], *no-restriction hypothesis* [31, 40], and *purification* [10, 22]. Thanks to the generality and suitability of this framework, various tasks of informational, cryptographic, and computational nature have been broadly investigated in theory-independent scenarios, e.g. signalling [15, 40, 21], broadcasting [4], teleportation [5], and bit commitment [44], among others [6, 33, 3, 34, 30, 45]. This approach has even been exploited in the attempt of clarifying the role of probabilities in QT. Most notably, the popular Spekkens' toy theory can be counted among the examples supporting an epistemic view of quantum states [47, 38, 7, 8].

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A parallel and prolific approach that has been developed in the recent years is based on category theory [2, 19]. The categorical framework allows one to provide a rigorous notion of compositionality, not only between systems or states, but also between operations. Furthermore, this perspective enables one to get a solid grasp on general theories (even non-probabilistic ones, e.g. theories of processes) relying on the powerful tool of *diagrammatic reasoning* and calculus [13, 28, 14, 26, 12, 16, 24]. The aforementioned features of GPTs can be suitably reformulated in the context of *symmetric strict monoidal categories*, lending themselves to further generalisation [9, 43, 41]. This line of thought eventually led to the achievement of a rigorous reconstruction of QT within both a categorical and an information-theoretic perspective [1, 28, 11, 22, 43, 42].

A first facet of our work consists in building a bridge between the two above-mentioned approaches in a constructive way. First of all, we provide the definition of an Operational Probabilistic Theory (OPT) resorting to the categorical framework [22, 25, 48]. We define an OPT as a braided strict monoidal category, where two appropriate classes of morphisms (states and effects, respectively from and to the identity object) are contained in convex sets—and thus endowed with a linear structure. Operations (or transformations) are then defined as linear mappings from states to states. The strict-monoidal structure can be legitimized by minimal requirements that a physical theory should have in terms of sequential and parallel composition of operations. On the other hand, the existence of a braiding is required in order to have a well-defined operational equivalence between the systems of multipartite states (more in general, between the different ways of associating both systems and operations). After showing how a complete OPT can be axiomatically constructed in terms of composition rules and local transformations, we focus our investigation on the most familiar class of theories: classical theories. This class, despite its simplicity, turns out to be not as predictable as it seems.

2 Classical non-locality

Stated in its weakest form, a *classical theory* is defined as a theory where the space of states for every system is a simplex [4]. This definition is sufficiently general not only to encompass standard classical probability theory, but also more generic and somewhat pathological scenarios, e.g. *noisy* classical theories, where there are states that cannot be reliably distinguished from any other. The rationale of a classical theory is that when an experimenter is given a set of states, he or she is also allowed to have available arbitrary mixtures of these states, by generating (classical) randomness. Indeed, the simplectic structure is the minimal convex structure comprising a finite set of states, regardless of their permutability, distinguishability, or copiability. In fact, *a* classical theory, if not *the standard* one, is taken to be the backbone of any probabilistic theory (let alone the fact that the interface between an experimenter and the physical systems is modelled as inherently classical). Yet, we show that a classical theory not only *may* admit entangled states, but that it *certainly* does if and only if the theory is not standard CT (or a noisy version of it). The definition of entangled state we stick to is the traditional one: a state which is not the convex combination of product states—equivalently, which cannot be prepared with local operations and classical communication—is entangled.

We define two important properties in the context of OPTs [22]: (i) *local discriminability*, stating that the products of single-partite observations are separating for states, namely for each pair of states there exists an effect which is the product of single-partite effects and gives different statistics on the two states; (ii) *atomicity of state-composition*, essentially stating that the composition of two pure states is pure. Both properties are shared by both CT and QT. We are then able to prove our first main result.

Theorem 1 *Let Θ be a classical theory. Then Θ admits entangled states if and only if Θ does not satisfy* *local discriminability.*

This first result relies on the traditional GPT framework [6]. Analysing the admissibility of entangled states for a classical theory in presence of atomicity of state-composition, we are also able to prove the following.

Theorem 2 *Let Θ be a classical theory. Then Θ admits entangled states if and only if Θ does not satisfy atomicity of state-composition.*

Interestingly, the proof of the latter result is grounded on the monoidal structure of an OPT. As combining the two theorems reveals, satisfying local discriminability is equivalent to satisfying atomicity of state-composition in the case of classical theories. In fact, CT is a theory where the states of every system form a simplex *and* local discriminability holds (the pure states being perfectly discriminable, in the noiseless case).

3 A simplicial theory with entanglement

We apply our construction to a concrete case, explicitly building an OPT from scratch. The theory features a number of interesting properties. It is simplicial and causal, nonetheless exhibiting entanglement. The entangled states states are the vertexes of the simplexes associated to composite bi-partite systems, and are not discriminable (not even probabilistically) via local measurements. Indeed, the theory must violate local discriminability, satisfying bi-local discriminability instead. This means that effects involving bi-partite observations are needed to discriminate between arbitrary states. This is a consequence of the particular composition rule for the simplexes associated to the systems. Every product state of pure states is the flat mixture of a pair of entangled (pure) states of the bi-partite system. However, the pure states of every system are reversibly permutable and perfectly discriminable. The theory exhibits no complementarity, being simplicial [35]. Clearly, purification is not satisfied. Interestingly, this theory shares with CT also the fact that identity operation is not atomic, and that *no-information without disturbance* does not hold.

The theory is also relevant from the point of view of axiomatics of probabilistic theories. In particular, it provides an explicit example that atomicity of state-composition is independent from (a) causality, (b) local discriminability, (c) purification, and (d) bi-local discriminability. CT satisfies axioms (a,b), QT (a,b,c), while all axioms (a,b,c,d) are shared by both Real Quantum Theory and Fermionic Theory [29, 23]. This helps to pave the way for an understanding of the axioms independence, with a view to the formulation of post-classical and post-quantum theories. In addition, from a conceptual perspective, this theory shows how a definition of purity proposed in general process-theoretic scenarios—in the absence of a probabilistic structure (see e.g. Refs. [9, 41, 20])—might fail in a concrete probabilistic context, turning out to be empty. Finally, due to its features, the theory is well-suited for secret-sharing, cryptographic, and secure-multiparty protocols.

4 Conclusions and future work

Via a systematic treatment involving the axiomatics, we are able to show that what has been long considered one the most striking departure from CT, i.e. entangled states, is indeed a ubiquitous feature shared by a vast class of classical theories—essentially by every classical theory which is not the standard one. The space of states is not sufficient to fully determine a theory, and compositionality plays a major role.

The complete theory we constructed serves as a proof of concept of our claim, proving the non-emptiness of the family of classical theories without local discriminability.

The theory we studied admit a local hidden-variable model. However, it suffices to give up permutability of pure states (this can be done keeping perfect discriminability intact), to end up with a theory which does not admit any local hidden-variable model whatsoever. First, it seems interesting to investigate the connections of the present work with Spekkens' toy theory [47]. In the context of general process theories, we observe the interesting stark contrast with the definition of *pure processes* provided e.g. in Ref. [41]. Finally, we believe that the intriguing relation with the line of work of e.g. Refs. [32, 39] is worth being subject of further exploration.

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