Phase space simulation method for quantum computation with magic states on qubits

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Abstract

We propose a method for classical simulation of finite-dimensional quantum systems, based on sampling from a quasiprobability distribution, i.e., a generalized Wigner function. Our construction applies to all finite dimensions, with the most interesting case being that of qubits. For multiple qubits, we find that quantum computation by Clifford gates and Pauli measurements on magic states can be efficiently classically simulated if the quasiprobability distribution of the magic states is non-negative. This provides the so far missing qubit counterpart of the corresponding result [V. Veitch, C. Ferrie, D. Gross, and J. Emerson, *New J. Phys.* **14**, 113011 (2012)] applying only to odd dimension. Our approach is more general than previous ones based on mixtures of stabilizer states. Namely, all mixtures of stabilizer states can be efficiently simulated, but for any number of qubits there also exist efficiently simulable states outside the stabilizer polytope. Further, our simulation method extends to negative quasiprobability distributions, where it provides amplitude estimation. The simulation cost is then proportional to a robustness measure squared. For all quantum states, this robustness is smaller than or equal to stabilizer robustness.

NB: This extended abstract is submitted in lieu of an arxiv submission to QPL 2019; however we do not wish to submit the work for publication to the proceedings.

1 Introduction

The question of how to mark the classical-to-quantum boundary dates back to almost the beginning of quantum theory. Ehrenfest's theorem [2] provides an early insight, and the Einstein-Podolsky-Rosen paradox [3] and Schrödinger's cat [4] are two early puzzles. The advent of quantum computation [5], [6] added a computational angle to this question: *When does it become hard to simulate a quantum mechanical computing device on a classical computer? Which quantum mechanical resource do quantum computers exploit to generate a computational speedup?*

For a certain model of quantum computation, so-called quantum computation with magic states (QCM) [7], both "traditional" indicators of quantumness, developed in the fields of quantum optics and foundations of quantum mechanics, and a computational indicator can be applied. The former are the negativity of a Wigner function [8,9], and the breakdown of non-contextual hidden variable models [10,11]. The latter is the breakdown of efficient classical simulation. All three indicators locate the classical-to-quantum boundary in the same spot [1,12].

Quantum computation with magic states operates with a restricted set of instructions, the Clifford gates. These are unitary operations defined by the property that they map all Pauli operators onto Pauli operators under conjugation. Clifford gates are not universal, and, in fact, can be efficiently classically simulated [13]. This shortcoming is compensated for by invoking the "magic" states, which are special quantum states that cannot be created by Clifford gates and Pauli measurements. Suitable magic states restore quantum computational universality; and in fact QCM is a leading paradigm for fault-tolerant universal quantum computation. In sum, computational power is transferred from the quantum gates to the magic states, and one is thus led to ask: *Which quantum properties give the magic states their computational power?*

One such property is negativity in the Wigner function. A quantum speedup can arise only if the Wigner function of the magic states assumes negative values. If, to the contrary, the Wigner function is positive, then the whole quantum computation can be efficiently classically simulated [1]. A positive Wigner function also implies a non-contextual hidden variable model; and, for $n \geq 2$ quantum systems, the reverse direction holds as well [12], [14]. The onset of negativity in the initial Wigner function, the appearance of contextuality, and the breakdown of efficient classical simulation by sampling all occur simultaneously. Wigner function negativity and contextuality of the magic states are thus necessary quantum computational resources in QCM.

The problem with qubits. However, there is a catch. The picture described so far only applies if the local Hilbert space dimension d is *odd*. This excludes the full n-qubit case, which arguably is the most important. Approaches to the full qubit case have been made, e.g. through the rebit scenario [15] and multi-qubit settings with operational restrictions [16,17], or by invoking a Wigner function over Grassmann variables [18], or multiple Wigner functions at once [19]. Common to all these approaches is that they fail to efficiently simulate the evolution under general Clifford gates and Pauli measurements (subsequently "Clifford circuits"), unlike in the case of odd d [1]. This is a critical shortcoming, as it means that these methods cannot efficiently simulate n-qubit QCM even in instances where magic states are absent. At the same time, Clifford circuits are efficiently simulable, for all d, by the stabilizer formalism [13]. The above approaches [15,19] do not fully subsume the stabilizer formalism, whereas the Wigner function method for odd d does.

The cause for the above difficulties is a structural difference between even and odd d, which comes to light through Mermin's square and star [11], two simple state-independent proofs of the Kochen-Specker Theorem [20] based on Pauli observables. In even dimension, where such contextuality proofs exist, for $n \geq 2$ every quantum state, even the completely mixed state, is contextual [12]. However, if contextuality is ubiquitous then it cannot be a resource. This problem does not arise for odd d, where state-independent contextuality proofs based on Pauli observables do not exist [21], [16,17].

Here we address the full n-qubit case of quantum computation with magic states, from the perspectives of the classical-to-quantum transition and quantum computational resources. Our goal is to reproduce for $d = 2$, as closely as possible, the relations between Wigner function, hidden variable model and efficient classical simulation existing in odd d. Central to our discussion is a novel quasiprobability function W defined for all local Hilbert space dimensions d. It has the following general properties:

1. For all n and d, W is Clifford-covariant and positivity-preserving under Pauli measurements.

2. For all n and d, the stabilizer formalism is contained as a special case. All stabilizer states can be positively represented by W, and efficiently updated under Clifford operations.

3. If the local Hilbert space dimension d is odd, then W reduces to the standard Wigner function [22], [23] for odd finite dimension.

4. For $d = 2$, for any given quantum state ρ, W_ρ is non-unique. The set of phase point operators corresponding to W is over-complete.
5. For $d = 2$ (as well as for d odd [1]), the present formalism goes beyond the stabilizer formalism. (i) For every number n of qubits there exist non-mixtures of stabilizer states which are positively represented by W. (ii) For any quantum state ρ, the 1-norm of the optimal W_ρ is smaller or equal than the robustness of magic $\mathfrak{R}_S(\rho)$.

The following properties of W for special values of n (and $d = 2$) are also worth noting. (a) The Eight-state model [24] is a special case of W, namely for $n = 1$. (b) For Mermin's square [11], the present simulation algorithm saturates the lower bound [25] on the memory cost of classical simulation. (c) Up to two copies of magic T and H states are positively represented by W.

Main results. The focus of this paper is the multi-qubit case, $d = 2$. Based on the quasiprobability function W for $d = 2$, we establish the following main results:

(I) The set of states positively represented by W is closed under Pauli measurement (Theorem 2).

(II) If a quantum state ρ has a non-negative function W_ρ and W_ρ can be efficiently sampled, then, for every Clifford circuit applied to ρ, the corresponding measurement statistics can be efficiently sampled (Theorem 4). In particular, this is the case for stabilizer states, hence our result generalizes the celebrated Gottesman-Knill theorem [13]. In this sense, $W \geq 0$ leads to efficient classical simulation of the corresponding quantum computation.

(III) For any number n of qubits there exist bound magic states, i.e., states that are not useful for magic state distillation but are outside the stabilizer polytope. The existence of such states has been established earlier for odd-dimensional qudits [1] but not for qubits[1].

(IV) For $d = 2$, the n-system phase space has a more complicated structure than in the case of odd d, reflecting the fact that the phase point operators are dependent. The points in generalized multi-qubit phase space are classified (Theorem 1).

The four properties (I) – (IV) are counterparts of analogous results for odd d, established earlier. To summarize, we find that both for odd d and $d = 2$ the fundamental indicator of the classical-to-quantum transition is a quasiprobability distribution (Wigner function in case of odd d). Negativity of this quasiprobability function indicates quantumness, as usual in quantum optics [9], and in addition it has the computational interpretation of being a precondition for quantum speedup.

2 The generalized phase space

In this section we introduce the generalized n-qudit phase space $\mathcal{V}$, for any local Hilbert space dimension d, and a quasi-probability distribution $W : \mathcal{V} \longrightarrow \mathbb{R}$ living on it.

We choose a phase convention for the Pauli operators,

$$T_a = e^{i\phi(a)} X(a_X) Z(a_Z), \ \ \forall a = (a_X, a_Z) \in V := \mathbb{Z}_d^{2n}, \tag{1}$$

subject to constraint on the function $\phi : V \longrightarrow \mathbb{R}$ that the resulting operators T_a satisfy $(T_a)^d = I$, for all $a \in V$. As a consequence of this condition, all eigenvalues of the operators T_a are of the form ω^k, $k \in \mathbb{N}$, with $\omega := \exp(2\pi i/d)$.

[1]Ref. [34] has reported the existence of "qubit bound states", referring to inputs for which output amplitudes of Clifford circuits can be estimated up to inverse-polynomial errors. This is however a non-standard notion of "bound magic state", which are inputs for which a *probabilistic* efficient classical simulation of Clifford circuits is possible [1]. This distinction is key, as the former task is often easy for quantum processes that cannot be simulated probabilistically [35].

We now proceed to the definition of the phase point operators. We consider a subset Ω of V, and a function $\gamma : \Omega \longrightarrow \mathbb{Z}_d$, both subject to additional constraints that will be specified in Definitions 2–4 below. The pair (Ω, γ) specifies a corresponding phase point operator A_Ω^γ,

$$A_\Omega^\gamma := \frac{1}{d^n} \sum_{b \in \Omega} \omega^{\gamma(b)} T_b, \tag{2}$$

with the constraint that

$$\omega^{\gamma(0)} T_0 = I. \tag{3}$$

When comparing Eq. (2) to the phase point operators of the previously discussed qudit [1], rebit [15] and restricted qubit [16, 17] cases, we note that the overall structure remains the same. In this case, the sets Ω are an additional varying parameter, and the phase space thereby becomes larger.

Based on the phase point operators A_Ω^γ of Eq. (2), we introduce the counterpart to the Wigner function that applies to our setting. The generalized phase space $\mathcal{V}$ consists of all admissible pairs (Ω, γ), to be specified below. Any n-qudit quantum state ρ can be expanded in terms of a function $W_\rho : \mathcal{V} \longrightarrow \mathbb{R}$,

$$\rho = \sum_{(\Omega,\gamma) \in \mathcal{V}} W_\rho(\Omega, \gamma) A_\Omega^\gamma. \tag{4}$$

The reason for imposing Eq. (3) is that it implies $\text{Tr} A_\Omega^\gamma = 1$, for all $(\Omega, \gamma) \in \mathcal{V}$. Hence, W defined in Eq. (4) is a quasiprobability distribution. As we see shortly, it generalizes the Wigner function [22] for odd-dimensional qudits to qubits.

We note that the quasiprobability distribution W_ρ is non-unique because the set of phase point operators of Eq. (2) is overcomplete.

Definition 1. An n-qubit quantum state ρ is *positively representable* if it can be expanded in the form of Eq. (4), with $W_\rho(\Omega, \gamma) \geq 0$, for all $(\Omega, \gamma) \in \mathcal{V}$.

The efficient classical simulation algorithm described below applies to positively representable quantum states ρ. The non-uniqueness of W_ρ helps with finding positive representations.

We now turn to the properties of admissible sets Ω and functions γ that constrain the phase space $\mathcal{V}$. To begin, we define a function β which encodes how translations on phase space compose,

$$T_a T_b = \omega^{\beta(a,b)} T_{a+b}, \;\; \forall a, b \in V \text{ with } [a, b] = 0. \tag{5}$$

We constrain Ω by the following definitions:

Definition 2. A set $\Omega \subset V$ is *closed under inference* if it holds that

$$a, b \in \Omega \, \wedge \, [a, b] = 0 \Longrightarrow a + b \in \Omega. \tag{6}$$

The motivation for this definition is that if T_a and T_b can be simultaneously measured, then the value of T_{a+b} can be inferred from the measurement outcomes, through relation (5). A consequence of the closedness under inference is that $0 \in \Omega$ for all admissible sets Ω.

Definition 3. A set $\Omega \subset V$ is *non-contextual* if there exists a value assignment $\gamma : \Omega \longrightarrow \mathbb{Z}_d$ that satisfies the condition

$$\gamma(a) + \gamma(b) - \gamma(a + b) = \beta(a, b), \;\; \forall a, b \in \Omega, \text{ and } [a, b] = 0. \tag{7}$$

To motivate the nomenclature, if the set $\Omega \subset V$ is non-contextual per the above definition, then it does not admit a parity-based contextuality proof [28]. Namely, Eq. (7) represents the constraints on non-contextual value assignments γ that result from the operator constraints Eq. (5). If these constraints can be satisfied, then there is no parity-based contextuality proof.

Definition 4 (Generalized phase space $\mathcal{V}$)**.** The set $\mathcal{V}$ consists of all pairs (Ω, γ) such that (i) Ω is closed under inference, (ii) Ω is non-contextual, (iii) $\gamma : \Omega \longrightarrow \mathbb{Z}_d$ satisfies Eq. (7), and (iv) Eq. (3) holds.

2.1 Maximal sets

For short, we call sets Ω that are closed under inference and non-contextual "admissible". The admissible sets Ω partially determine the points in phase space, and it is thus desirable to eliminate possible redundancies among them. It turns out that only the "maximal" sets Ω need to be considered for $\mathcal{V}$.

Definition 5. An admissible set $\Omega \subset V$ is maximal if there is no admissible set $\tilde{\Omega} \subset V$ such that $\Omega \subsetneq \tilde{\Omega}$.

We denote by $\mathcal{V}_M$ the subset of $\mathcal{V}$ constructed only from the maximal admissible sets Ω. Then, any quantum state ρ has expansions like Eq. (4), but with $\mathcal{V}$ replaced by $\mathcal{V}_M$. If one of those expansions is non-negative, then we say that ρ is positively representable w.r.t. $\mathcal{V}_M$.

Lemma 1. *For any n and d, a quantum state ρ is positively representable w.r.t. $\mathcal{V}$ if and only if it is positively representable w.r.t. $\mathcal{V}_M$.*

From the perspective of positive representability, we may therefore shrink $\mathcal{V}$ to $\mathcal{V}_M$ without loss. We make use of this property when discussing the case of odd d right below, and in the classification of admissible sets Ω for the multi-qubit case.

2.2 Qudits of odd local dimension

This is the only place in the present paper where we consider the case of odd d. The purpose of this section is to show that if d is odd then the generalized phase space $\mathcal{V}$ reduces to the standard phase space $V = \mathbb{Z}_d^{2n}$, and the quasiprobability function W becomes the standard Wigner function [22] for odd finite-dimensional systems.

The Wigner function [22] in odd local dimension is the starting point for the analysis of contextuality as a resource in quantum computation with magic states [12], [27]; and since the present W is defined for both odd and even d, the present discussion of contextuality for the qubit case ($d = 2$) can relate to the earlier one. In fact, by way of the odd-dimensional relative, the quasiprobability function W is a descendant of the original Wigner function [8].

If d is odd then the whole set V is admissible. First, V is closed under inference by definition. And second, it is known that in odd dimension Pauli observables have non-contextual deterministic value assignments [21], [16, 17]. These yield the functions γ, satisfying the condition Eq. (7). V is thus non-contextual.

V is furthermore the single maximal set, and, with Lemma 1, the only admissible set that needs to be considered for the phase space $\mathcal{V}$. Hence, the phase point operators are

$$A_V^\gamma = \frac{1}{\sqrt{|V|}} \sum_{a \in V} w^{\gamma(a)} T_a,$$

with the functions γ satisfying (3) and Eqs. (7). The former condition ensures that the identity operator appears with weight $1/\sqrt{|V|}$ in the expansion (real and positive). The latter condition has d^{2n} solutions for the functions γ if d is odd [14]. For a suitable choice of ϕ in Eq. (1), it holds that $\beta \equiv 0$ (odd d only). The solutions for γ then form a vector space

$$\mathcal{V} = \mathbb{Z}_d^{2n} \quad \text{(for odd } d\text{)}.$$

The remainder of this paper is about local Hilbert space of dimension $d = 2$.

2.3 Structure of the qubit phase space

The notion of "closed and non-contextual sets Ω" is central for the definition of the qubit phase space. So what is the explicit form if the sets? Here characterize their general structure.

Lemma 2. *For n qubits, consider an isotropic subspace $\tilde{I} \subset V$ of dimension $n-m$, with $m \leq n$, and $2m+1$ elements $a_k \in V$ that pairwise anti-commute but all commute with $\tilde{I}$. Denote $I_k := \langle a_k, \tilde{I} \rangle$ for $k = 1, .., 2m+1$. For any number n of qubits, the sets*

$$\Omega = \bigcup_{k=1}^{2m+1} I_k \tag{8}$$

are non-contextual and closed under inference.

Theorem 1. *All maximal admissible sets Ω are of the form Eq. (8).*

The eight-state model and bound magic states. We note here that the qubit stabilizer formalism and the Eight-state model [24] are both contained as a special case. Namely, $m = 0$ for the stabilizer formalism and $n = 1$ for the Eight-state model. Also note that there exist states which are not mixtures of stabilizer states which are positively represented. For example, any n-qubit state of the form $\Psi = \rho_1 \otimes |stab\rangle\langle stab|_{2,\dots,n}$ where ρ_1 is any one-qubit state and $|stab\rangle_{2,\dots,n}$ is a $n-1$-qubit pure stabilizer state, is positively representable by phase points corresponding to sets Ω with parameter $m = 1$.

3 Properties of the quasi-probability function W

3.1 State update under measurement

The purpose of this section is to describe the update of the phase point operators A_Ω^γ under Pauli measurement, and to show that positivity of the quasiprobability distribution W is preserved. These results are stated as Lemma 3 and Theorem 2 below. Note that the update of W under Clifford unitaries is not needed from the perspective of classical simulation, because the Clifford unitaries may always be propagated out of the circuit past the final measurement.

Theorem 2. *For any $n \in \mathbb{N}$, the set $\mathcal{P}_n$ of positively representable n-qubit quantum states is closed under Pauli measurement.*

It is easily checked that all probabilistic mixtures of n-qubit stabilizer states are in $\mathcal{P}_n$. Further it is obvious that the stabilizer polytope is closed under Pauli measurement. However, the point is that the sets $\mathcal{P}_n$ contain more than mixtures of stabilizer states. This holds already in the single-qubit case, $n = 1$.

To describe the dynamics under measurement, we need to set up some further notation. For every set Ω we introduce the derived set $\Omega \times a$. Denoting $\text{Comm}(a) := \{b \in V | [a,b] = 0\}$ and $\Omega_a := \Omega \cap \text{Comm}(a)$,

$$\Omega \times a := \Omega_a \cup \{a + b | \, b \in \Omega_a\}, \quad \forall a \notin \Omega. \tag{9}$$

Likewise, we define an update on functions γ invoking the measurement outcome s_a of an observable T_a, namely $(\cdot) \times s_a : (\gamma : \Omega \longrightarrow \mathbb{Z}_2) \mapsto (\gamma \times s_a : \Omega \times a \longrightarrow \mathbb{Z}_2)$. We define this update only for $(\Omega, \gamma) \in \mathcal{V}$, and only for $a \notin \Omega$ [2]. The updated function $\gamma \times s_a : \Omega \times a \longrightarrow \mathbb{Z}_2$ is given by

$$\gamma \times s_a(b) := \gamma(b), \qquad \forall b \in \Omega_a, \tag{10a}$$

$$\gamma \times s_a(b) := \gamma(a+b) + s_a + \beta(a,b), \qquad \forall a + b \in \Omega_a. \tag{10b}$$

These definitions are needed to state and derive the update rule for phase point operators under Pauli measurement, which are as follows.

Lemma 3. *Denote the projectors $P_a(s_a) := (I + (-1)^{s_a} T_a)/2$, and be A_Ω^γ a phase point operator defined through Eq. (2), with $(\Omega, \gamma) \in \mathcal{V}$ satisfying the conditions of Definition 4. Then, the effect of a measurement of the Pauli observable T_a with outcome s_a on A_Ω^γ is*

$$P_a(s_a) A_\Omega^\gamma P_a(s_a) = \delta_{s_a, \gamma(a)} \frac{A_\Omega^\gamma + A_\Omega^{\gamma + [a,\cdot]}}{2}, \qquad \text{if } a \in \Omega, \tag{11a}$$

$$P_a(s_a) A_\Omega^\gamma P_a(s_a) = \frac{1}{2} A_{\Omega \times a}^{\gamma \times s_a}, \qquad \text{if } a \notin \Omega. \tag{11b}$$

3.2 Covariance of W under Clifford gates

Be Cl_n the n-qubit Clifford group. It acts on the n-qubit Pauli operators via

$$h(T_a) := h T_a h^\dagger = (-1)^{\Phi_h(a)} T_{ha}, \;\; \forall h \in \text{Cl}_n.$$

This implies an action of the Clifford group on the phase point operators A_Ω^γ, which in turn induces an action on the sets Ω and the functions γ, via

$$h(A_\Omega^\gamma) = \frac{1}{2^n} \sum_{a \in \Omega} (-1)^{\gamma(a)} h(T_a) = \frac{1}{2^n} \sum_{b \in \Omega'} (-1)^{\gamma'(b)} T_b.$$

Therein, the set Ω' is defined as $\Omega' := \{ha, \; a \in \Omega\}$, and the function $\gamma' : \Omega' \longrightarrow \mathbb{Z}_2$ is given by

$$\gamma'(ha) := \gamma(a) + \Phi_h(a), \; \forall a \in \Omega.$$

Henceforth we denote Ω' as $h \cdot \Omega$ and γ' as $h \cdot \gamma$, to emphasize the dependence on $h \in \text{Cl}_n$.

We now have the following result.

Theorem 3. *$\mathcal{V}$ is mapped to itself under Cl_n, and the quasiprobability function W transforms covariantly. That is, if the state ρ can be descried by W_ρ through Eq. (4), then for any $h \in Cl_n$ the state $h\rho h^\dagger$ can be described by a quasiprobability function $W_{h\rho h^\dagger}$ defined by*

$$W_{h\rho h^\dagger}(\Omega, \gamma) := W_\rho(h^{-1} \cdot \Omega, h^{-1} \cdot \gamma).$$

[2] The definitions of $\Omega \times a$ and $\gamma \times s_a$ can without modification be extended to $a \in \Omega$. However, in that case the function values $\gamma \times s_a(b)$ can be determined both through Eq. (10a) and (10b), and we need to check consistency. These inferences are indeed consistent, as a consequence of Eq. (7). Since we do not need the case of $a \in \Omega$ subsequently, we skip the details of the argument.

Classical simulation algorithm

1. Draw a sample $(\Omega, \gamma) \in \mathcal{V}$ according to the probability distribution W_ρ representing the initial quantum state ρ.
2. For the observables $T_{a_1}, T_{a_2}, .., T_{a_N}$ measured in this sequence, repeat the following steps.

   For the i-th measurement, set $a := a_i$.

   If $a \in \Omega$ then Ω is unchanged. Output the value $s_a = \gamma(a)$. Flip a coin.

   $$\begin{array}{ll} \text{if "heads"} & \text{then } \gamma \longrightarrow \gamma, \\ \text{if "tails"} & \text{then } \gamma \longrightarrow \gamma + [a, \cdot]. \end{array}$$

   If $a \notin \Omega$ then $\Omega \longrightarrow \Omega \times a$. Flip a coin.

   $$\begin{array}{ll} \text{if "heads"} & \text{then } s_a = 0, \\ \text{if "tails"} & \text{then } s_a = 1. \end{array}$$

   Output this value s_a. Update $\gamma \longrightarrow \gamma \times s_a$, through Eq. (10).

Table 1: Classical simulation algorithm for sampling from the joint probability distribution of a sequence of Pauli measurements on a positively represented initial quantum state.

4 Classical simulation algorithm

We now turn to the question of how hard it is to classically simulate the outcome statistics for a sequence of Pauli measurements on an initial quantum state. In this regard, we show that if the initial quantum state is positively represented and the corresponding probability distribution W can be efficiently sampled from, then the statistics of the measurement outcomes can be efficiently simulated. Note that all Clifford gates in the quantum algorithm are redundant, since they can be propagated forward past the last measurement. To simplify the statement of the simulation algorithm, we therefore do not consider Clifford unitaries in the quantum algorithm but only the Pauli measurements.

The classical simulation procedure described below in Table 1 outputs one sample from the joint probability distribution $p(s_{a_1}, s_{a_2}, .., s_{a_N})$ of outcomes corresponding to a sequence of measurements of Pauli operators $T_{a_1}, T_{a_2}, .., T_{a_N}$ (T_{a_1} is measured first, T_{a_N} last). If more than one sample are desired, the procedure is just repeated. We note that the observables can be chosen at runtime. I.e., it is not necessary for the simulation algorithm that a measurement sequence is committed to at the beginning. As a special case of this, the measured observables may depend on earlier measurement outcomes.

We consider a sequence of Pauli measurements on an initial n-qubit state ρ that is positively representable, such that the classical simulation algorithm of Table 1 applies. We then have the following result.

Theorem 4. *The algorithm of Table 1 is correct, and, if the initial probability distribution W_ρ can be efficiently sampled from, it is also computationally efficient.*

We outline the intuition why the above algorithm works in polynomial time. By Theorem 1, all sets Ω split into orbits Ω_i, each of which is a set of labels of some Pauli operators $\{(-1)^{\gamma(a)}T_a, a \in \Omega_i\}$ which defines a stabilizer group. Thus, we can regard A_Ω^γ as a union of $O(n)$ stabilizer groups. The update rules in algorithm 1 carry out measurement updates of these stabilizer groups under Pauli measurements in the stabilizer formalism [13]. Computing $s_a = \gamma(a)$ corresponds to reading the phase of a stabilizer operator T_a^γ. If each orbit is described in term of a small $O(n)$ set of stabilizer generators, the simulation reduces efficiently to tasks that admit polynomial time algorithms via the Gottesman-Knill theorem [13].

5 Probabilistic hidden variable model

In the qudit case [1], a positive Wigner function is equivalent to a non-contextual hidden variable model (HVM) [12]. This begs the question of which similar relations exist in the present multi-qubit case, if any. As a consequence of Mermin's square, for $n \geq 2$ all quantum states—even the completely mixed state—are contextual [12]. In this situation, what is the interplay between contextuality and classicality?

5.1 Description of the HVM

Our HVM consists of a triple $(\Lambda, \{h^\lambda\}, p_\lambda)$ where $\Lambda = \mathcal{V}$ or $\mathcal{V}_M$, h^λ is a compatible family of distributions on the set of outcomes on contexts and p_λ is a probability distribution on the set Λ of hidden variables. For each $\alpha = (\Omega, \gamma)$ we define h^α by

$$h_I^\alpha(s) = \mathrm{Tr}(P_s A_\Omega^\gamma). \tag{12}$$

Therein, I is any isotropic subspace, $s : I \to \mathbb{Z}_2$ is a function, and P_s is the projector corresponding to the outcome. Note that $P_s = 0$ if $ds \neq \beta$.

It is useful to state the probability distributions $h_I^\alpha(\cdot)$ in their explicit form.

Lemma 4. *Let P_s denote the projector corresponding to the non-contextual value assignment $s : I \to \mathbb{Z}_2$. Then we have*

$$h_I^{(\Omega,\gamma)}(s) = \frac{|I \cap \Omega|}{|I|}\delta_{s|_{I\cap\Omega},\gamma|_{I\cap\Omega}}. \tag{13}$$

From Eq. (13) we see that the value assignments in our HVM are generally probabilistic; only in the special case of $I \subset \Omega$ they become deterministic. Further we observe that the $\{h_I^\alpha\}$ form compatible families,

$$h_I^\alpha|_{I\cap I'} = h_{I'}^\alpha|_{I'\cap I},\ \forall I, I',\ \forall \alpha \in \mathcal{V}.$$

5.2 Contextuality coexisting with classicality

Is the probabilistic HVM described by Eq. (13) contextual or non-contextual?—Multiple definitions of contextuality have been proposed, and the following statements hold:

- The HVM of Eq. (12) is preparation contextual but measurement-non-contextual, in the sense of Spekkens [31].
- This model is contextual in the sense of Abramsky and Brandenburger [26].

At the same time, our model shows elements of classicality. It is naturally casted as an ontological model (see, e.g., [30]), wherein states are probability distributions over the "ontic" states A_Ω^γ, with associated effects (12). We can regard the model classical in the statistical mechanical sense.

Namely, for any quantum state ρ with quasiprobability distribution W_ρ, for all contexts I it holds that the probability distribution p_I of measurement outcomes is

$$p_I(s) = \sum_{\alpha \in \mathcal{V}} W_\rho(\alpha) h_I^\alpha(s). \tag{14}$$

If $W_\rho \geq 0$, then it can be considered as a probability distribution for finding the system in a state α in phase space.

6 Classical simulation for $W_\rho < 0$

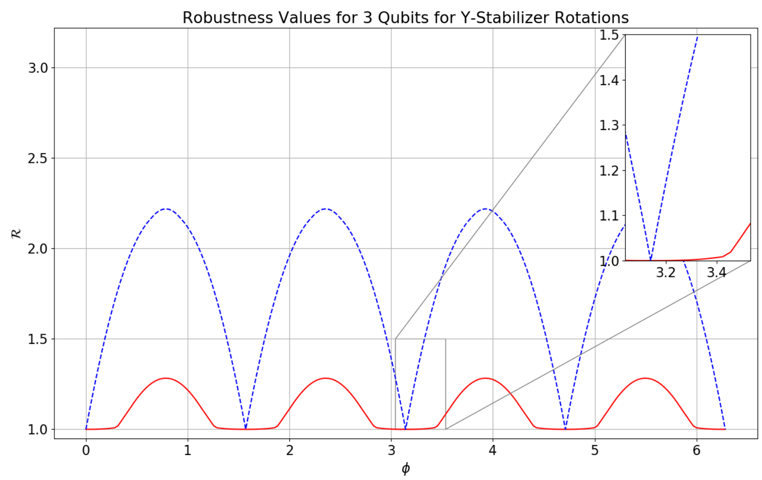

Figure 1: Robustness values plotted for the state $\left(\frac{e^{i\phi/2}|0\rangle + e^{-i\phi/2}|1\rangle}{\sqrt{2}}\right)^{\otimes 3}$. The full line is the robustness $\mathfrak{R}$ of Eq. (15), and the dashed line the stabilizer robustness (a.k.a. robustness of magic) $\mathfrak{R}_S$. Inset: change in the robustness as it moves away from the stabilizer state at $\phi = \pi$.

In the absence of a true probability distribution, a standard problem of interest becomes to estimate outcome probabilities for sequences of measurements given a quasiprobability function representing the initial state. An established method for this is [33], utilizing the Hoeffding bound. In this amplitude estimation the number N of samples required to estimate the output probability distribution up to an error ϵ scales as $N \sim \mathcal{M}^2/\epsilon^2$, where $\mathcal{M}$ is a measure of the negativity contained in the quantum process. This negativity is quantified by the robustness $\mathfrak{R}$, defined as

$$\mathfrak{R}(\rho) := \min_{W |\, \langle \mathcal{A}, W \rangle = \rho} \|W\|_1, \tag{15}$$

with $\langle \mathcal{A}, W \rangle := \sum_{\alpha \in \mathcal{V}} W_\alpha A_\alpha$. The algorithm of Pashayan et al. [33], when applied to our setting, says that the number N of samples required to estimate the output probability scales as

$$N \sim \frac{\mathfrak{R}(\rho_{\text{init}})^2}{\epsilon^2}.$$

Thus, the robustness $\mathfrak{R}(\rho_{\text{init}})$ of the initial state ρ_{init} is the critical parameter determining the classical hardness of probability estimation.

The same relation, with the robustness $\mathfrak{R}$ replaced by the stabilizer robustness $\mathfrak{R}_S$ holds for the classical simulation based on quasiprobability distributions over stabilizer states [29]. It is therefore if interest to establish relations between these two robustnesses.

Lemma 5. *For any number n of qubits, the phase space robustness $\mathfrak{R}(n)$ and the stabilizer robustness $\mathcal{R}_S(n)$ are related via*

$$\mathfrak{R}(n) \leq \mathfrak{R}_S(n) \leq (2n+1)\,\mathfrak{R}(n). \tag{16}$$

Thus, the present robustness $\mathfrak{R}$ is never larger than the stabilizer robustness, but can only be moderately smaller. An exemplar of the differences between the phase space robustness and the stabilizer robustness is shown for three qubits in Figure 1.

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