Irreversibility and non-classicality in a single system game

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The CHSH-Bell inequality [1, 2] is considered one of the most important discoveries in physics of the 20th century. It identifies limits on the correlations achievable in an experiment, when measurements are made on space-like separated systems, for classical physics and any local realistic theory. In the era of quantum information theory, this classic experiment has been recast as the CHSH game [3], a game between two players, Alice and Bob, who are separated and unable to communicate with each other, and a referee who asks them binary questions. They win if the sum of their answers is equal to the product of the questions (arithmetic modulo 2). The success probability of the game depends on the strength of correlations between the players. The CHSH-Bell inequality represents the best average success probability for two *classical* players to win the game. This upper bound is known as the Bell bound. Strikingly, quantum mechanics allows this inequality to be violated, achieving correlations impossible in any classical theory. However, the quantum maximum success probability is itself bounded, and the bound is known as the Tsirelson's bound [4]. The latter has been the topic of intense research in recent years. One particular aim has been to understand the origin of Tsirelson's bound, as it may shed light on the intrinsic properties of quantum mechanics.

The CHSH game can be generalized to mod$q$ arithmetic in the CHSH$_q$ game, which has been studied in [5, 6, 7, 8]. Naturally, a key focus of these studies has been to find the Bell bound and Tsirelson bound for these games. However, success has been limited. Upper bounds on the Tsirelson bound given by a precise mathematical expression have been provided in [8] when q is a prime or prime power, but these are not known to be tight. Moreover, numerical analysis on lower and upper bounds suggest different values [7]. The CHSH game is of great importance because the sensitivity of its optimal success probability depending on the underlying physical model gives us a tool to distinguish different types of theories experimentally, and allows us to test nature. It also reveals insights into a non-classical feature of quantum mechanics (known colloquially as "non-locality"), which has proven to be a resource for quantum technologies, such as device independent cryptography [9].

Other computational protocols showing similar features to the CHSH game, where strategies based on quantum mechanics perform better than classical strategies, exist [10, 11, 12, 13, 14]. In particular, in quantum random access codes (QRACs) [10], where Alice encodes m bits in $n < m$ information carriers to communicate Bob the value of one of the bits (randomly chosen), the optimal classical and quantum strategies are closely related to the ones used in the CHSH protocol and provide the same bounds.

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Inspired by these works, we here propose and investigate a single-system protocol, which is a simple single-player variant of the CHSH game. To play the game, the player has a system in a fixed initial state, two gates controlled by classical input bits and a measurement at the end (figure 1). The task is to output a non-linear function – the product – of the input bits in mod 2 arithmetic. Due to its similarity with the CHSH game we call it the *CHSH* game*. However, unlike the CHSH game that involves two space-like separated parties, the CHSH* game cannot involve any non-locality argument to explain the computational advantages. Similarly, it does not show contextuality in its standard formulations [15, 16] – the other candidate that is typically used to explain the quantum computational speed-up [17] – as there are no contexts as usually defined.

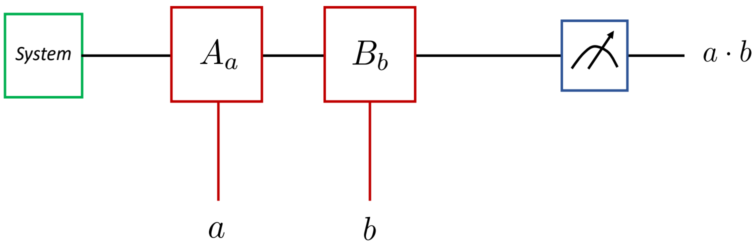

Figure 1: **Single-system protocol.** An initial system is subjected to controlled transformations, with control bits a and b, respectively, and then measured. The goal is to maximize the probability that the value of the output is the product of the values of the input bits.

We study the probability of success of the CHSH* game in different settings. We first show that, when the player applies unitary dynamics and projective measurements on a qubit system, the maximum probability of success of the game is equal to Tsirelson's bound; this is proven via an explicit mapping from the strategies in the CHSH* game to the strategies in CHSH game. We then illustrate that the game is sensitive to a broad range of properties of the system used, specifically whether the system is quantum or classical, what is the set of operations allowed to the player (namely reversible versus irreversible and Clifford versus non-Clifford) and what is the dimension of the system.

More precisely, we demonstrate that the Bell bound holds for classical reversible strategies and quantum strategies involving only Clifford computation, while the possibility of performing irreversible computation allows one to win the game with certainty. We also prove that non-Clifford gates play a crucial role in order to obtain better than classical performance in quantum computation. We show that, under the assumption of reversible transformations, the CHSH* game acts as a dimensional witness, since any initial state of dimension $d > 2$ can in principle win the game with certainty. However, the restriction to reversible operations is not a limitation. In accordance with Landauer's principle [18], implementing irreversible transformations at the microscopic level requires ancillary bits which must then be erased. The presence of exactly these hidden ancillary bits is detected by our protocol.

We focus on the role of irreversible computation as a source of computational advantage by considering the entropic costs of the erasure associated with the game, which is a powerful tool for increasing the winning probability. The lack of such an erasure operation in unitary quantum mechanics is a barrier to winning the game deterministically. Tsirelson's bound can be seen as arising from the absence of irreversible transformations and the limited ability of quantum strategies with unitary gates and projective measurements to simulate erasure. Nevertheless, quantum strategies can still perform better than the classical reversible strategies. What is the reason for that?

In addition to the content of the main article referenced at the beginning, we here add some preliminary results regarding the source of quantum computational advantages in the protocol and its relation with the erasure of information [20]. More precisely, we focus on contextuality. We have already mentioned that standard notions of contextuality do not apply here, as the contexts considered in Kochen-Specker contextuality [15] – different sets of commuting projectors – cannot be found, because the protocol involves only one fixed measurement at the end. Similarly, the contexts considered in Spekkens' notion of contextuality [16] – different decompositions of a quantum channel, of a quantum state or of a measurement element – are not present . However, by appealing to theorem 1 in [21], sequential transformation contextuality (with the assumption of $l2-$ontology) is necessary to enable quantum advantage over classical reversible resources in the CHSH* game. Sequential transformation contextuality refers to the fact that the same transformation in different sequences of transformations is not represented by the same probability distribution at the ontological level. The assumption of $l2-$ontology means that the underlying ontic space is assumed to be $\mathbb{Z}_2^n$ for some $n \in \mathbb{N}$ and transformations are mod2$-$ linear. It is argued in [21] that this is a natural assumption for protocols like the CHSH* game. We can quantify the contextuality involved in the game by using the contextual fraction [22], which obeys the following relation ([21, Theorem 1]),

$$p_{\text{fail}} \geq (1 - CF)\nu(f), \tag{1}$$

where $CF \in [0, 1]$ denotes the contextual fraction, p_{fail} is the probability of failure and $\nu(f)$ is the nonlinearity of the function $f = a \cdot b \bmod 2$. The nonlinearity of a Boolean function $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ is defined, in general, as the distance between the function f and the closest $\mathbb{Z}_2$-linear function $g : \mathbb{Z}_2^n \to \mathbb{Z}_2$,

$$\nu(f) = \min_g \{d(f, g) \mid g : \mathbb{Z}_2^n \to \mathbb{Z}_2 \ \text{ is } \ \mathbb{Z}_2\text{-linear}\}.$$

We recall that the average distance between two boolean functions $f, g : \mathbb{Z}_2^n \to \mathbb{Z}_2$ is defined as $d(f, g) = \frac{1}{2^n}|\{i \in 2^n | f(i) \neq g(i)\}|$, *i.e.* the fraction of the number of inputs for which the two functions differ.

Given the above technical notions and considering that erasing information, *i.e.* allowing irreversibility, is a source of computational advantage in the CHSH * game, we also provide a definition of the fraction of erasure (in terms of the unit – $k \log_2 2$ – of bit erased) associated to a given strategy that performs better than the optimal reversible classical strategy in the CHSH* game. The idea behind this notion is that a probability of success greater than the Bell bound can always be obtained with a strategy involving a bit, reversible gates and a gate that performs partial erasure of information. We define the Landauer's erasure, $LE \in [0, 1]$, associated to the probability of success p_{suc} of a strategy for the task of computing the non-linear function f in the CHSH* game, as

$$LE = \frac{p_{\text{suc}} - p_{\text{suc}}^{rev}}{\nu(f)}, \tag{2}$$

where p_{suc}^{rev} denotes the probability of success of the optimal reversible classical strategy, *i.e.* the Bell bound, which amounts to 0.75. For example, when considering a strategy that achieves the Tsirelson bound, which amounts to $p_{\text{suc}} = \cos^2(\frac{\pi}{8})$, the Landauer's erasure is $LE = \frac{\cos^2(\frac{\pi}{8}) - 0.75}{0.25} = 0.41$. The definition (2) actually provides a lower bound on the amount of erasure needed to perform with a certain probability of success p_{suc} and it can be rearranged in the relation

$$p_{\text{fail}} \geq (1 - LE)\nu(f), \tag{3}$$

considering that $p_{\text{suc}}^{rev} = 1 - \nu(f)$ and that the probability of failure is $p_{\text{fail}} = 1 - p_{\text{suc}}$.

We notice that the Landauer's erasure and the contextual fraction obey the same relations (equations (3) and (1)). This suggestive analogy may point at an interpretation of the phenomenon of contextuality as arising from a physical mechanism of erasure of information. It would be interesting to see if the previous analysis can be extended to other protocols like the 2−bits parity oblivious multiplexing [11]. This is a work in progress and we leave the study of the relation between irreversibility and non-classicality, in particular in the form of contextuality, for future research.

We conclude by briefly comparing our results in the article with related results in previous works. We note a similarity between the optimal unitary strategy for the CHSH* game and $2 \to 1$ QRACs [10] (and 2−bits parity oblivious multiplexing [11]). The latter have also been proposed as dimensional witnesses [19]. It is therefore important to emphasize the differences between RACs and the CHSH* game. The CHSH* game is able to detect the hidden information needed to implement irreversible gates. However, irreversible gates provide no advantage for the implementation of RACs. This means that a dimensional witness based on the RAC protocol will be blind to this kind of hidden information. Following Landauer's approach, we assert that the ability to detect irreversible dynamics should be an important desideratum for quantum dimensional witnesses. This has not been considered in prior work.

Finally, we conjecture our results to hold also for the generalization of the protocol to $\bmod q$ arithmetics. We support this by examining the $q = 3$ case in the single system scenario, for which we show the validity of the Bell bound and we further provide a strategy to achieve Tsirelson's bound. The validity of this conjecture may open the way to easier approaches for deriving Tsirelson's bounds in $\bmod q$ arithmetics, by using our single-system protocol as a tool for proving tightness.

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