Strongly symmetric convex bodies are Jordan algebra state spaces (Abstract for QPL 2019 talk)

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In [1], the normalized state spaces of simple finite-dimensional Euclidean Jordan algebras, and the simplices, were characterized as the unique finite-dimensional compact convex sets satisfying three properties: spectrality, strong symmetry, and the absence of higher-order interference. In the paper abstracted here, which is available on the arxiv at:`https://arxiv.org/abs/1904.03753` (direct download at: `https://arxiv.org/pdf/1904.03753`) we show that the same class of convex compact sets is characterized by the first two of these properties:

Theorem 1. *A finite-dimensional convex compact set is spectral and strongly symmetric if and only if it is a simplex or affinely isomorphic to the space of normalized states of a simple Euclidean Jordan algebra.*

By adding one or two additional assumptions, such as energy observability [1] (or the closely related *Connes orientation* [2, 3] or *dynamical correspondence* [3]) or the existence of tomographically local composites of any pair of state spaces in the theory [4] one can use this theorem to obtain simple characterizations of the usual mixed-state spaces of finite-dimensional quantum mechanics over the complex field, i.e. the sets of $n \times n$ Hermitian density matrices.

We use some notions associated with the framework of *general probabilistic theories* (GPTs), in which convex compact sets Ω are interpreted as normalized state spaces of physical systems, and indexed sets $\{e_i\}$ of *effects* (affine functionals on Aff Ω whose values on elements of Ω lie in $[0,1] \subseteq \mathbb{R}$, i.e., are probabilities) summing to the constant functional with value 1, are interpreted as measurements, with $e_i(\omega)$ giving the probability of the i-th outcome of the measurement, when made on a system prepared in state ω. Ω is usually embedded in a vector space V of one dimension greater than that of Ω, such that the affine hyperplane Aff Ω does not contain the origin; the cone in V with base Ω is denoted V_+.

A set of states $\omega_i, i \in \{1,...,k\}$ in Ω is called *perfectly distinguishable* if there is a measurement $\{e_i\}, i \in \{1,...,k\}$ such that $e_i(\omega_j) = \delta_{ij}$. A sequence $\omega_1,...,\omega_k$ of perfectly distinguishable *pure* states is called a *frame*, or a k-frame if we wish to specify its cardinality. A frame is called *maximal* if it is not a subsequence of any other frame. A convex compact set Ω of finite dimension n is called *spectral* (or, as in [1], *classically decomposable*) if, for each point $\omega \in \Omega$, there is some frame whose convex hull contains ω. It is called *strongly symmetric* if, for each $k \in \{1,...,n+1\}$, the group Aut Ω of affine maps taking Ω *onto* itself (its *automorphism group*) acts transitively on the set of k-frames. (The set of k-frames may be empty for many values of k in which case, trivially, Aut Ω acts transitively on it.)

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Finite-dimensional Euclidean Jordan algebras (EJAs) were introduced by Pascual Jordan around 1932 [5], as a possible algebraic setting for the formalism of quantum theory. The notion abstracts properties of the symmetrized product $A \bullet B := (AB + BA)/2$ on the complex Hermitian matrices, which are the observables[1] of a finite-dimensional quantum system. Unlike the ordinary associative matrix product, this preserves Hermiticity. A Jordan algebra is a real algebra (i.e., a real vector space V equipped with a commutative bilinear product $\bullet : V \times V \to V$) that satisfies a special case of associativity, the *Jordan property*: $a^2 \bullet (a \bullet b) = a \bullet (a^2 \bullet b)$, where $a^2 := a \bullet a$. It is *Euclidean* if it is possible to introduce an inner product $(\cdot\,,\cdot) : V \times V \to \mathbb{R}$ that is "associative", i.e. $(a \bullet b, c) = (a, b \bullet c)$.

The set of squares in a Euclidean Jordan algebra V is a convex cone V_+ whose linear span is V. It is pointed and closed so it has compact convex bases, all of which are affinely isomorphic. The associative inner product $(.\,,.)$ may be chosen so that the cone is self-dual with respect to it. The base consisting of V_+ intersected with the hyperplane $\{x : (e,x) = 1\}$, where e is the Jordan unit, is the *normalized state space* of the Jordan algebra. The linear form $(e,.)$ is known as the *trace*.

Jordan, von Neumann and Wigner [6] classified the finite-dimensional simple EJAs. (The other finite-dimensional EJAs are finite products of these.) They are precisely the $n \times n$ self-adjoint matrices with entries in $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ and the 3×3 octonionic self-adjoint matrices, with symmetrized matrix multiplication $x \bullet y = (xy + yx)/2$ as Jordan product in each case, and the *spin factors* $\mathbb{R}^n \oplus \mathbb{R}$ for every $n \geq 1$, with product $(\mathbf{x}, s) \bullet (\mathbf{y}, t) = (t\mathbf{x} + s\mathbf{y}, \langle \mathbf{x}, \mathbf{y} \rangle + st)$. In the matrix cases, the cones of squares are the positive semidefinite (PSD) matrices, and the normalized state spaces are the unit-trace PSD matrices; for the spin factors, they are (spherical, i.e. constant-radius) balls.

The main tools we use in proving Theorem 1 are (1) the structure theory of spectral strongly symmetric convex sets from [1], especially the fact that the cone V_+ over a strongly symmetric spectral convex compact set Ω is *perfect*, i.e V can be equipped with an inner product such that each face F of V_+ (including V_+ itself) is a self-dual cone with respect to the restriction of the inner product to the linear span of F, and (2) the characterization of *regular* convex bodies by Madden and Robertson [7], building on [8].

A *flag* Φ of a compact convex set Ω is a sequence F_i of nonempty exposed faces of the set, such that $F_i \subsetneq F_{i+1}$. It is *maximal* if it is not a proper subsequence of any other flag. Ω is *regular* if its affine automorphism group K acts transitively on the set of its maximal flags. In [8] such sets are shown to be determined by the group K and a convex polytope $\pi(\Omega)$, which we call the "Farran-Robertson polytope", in a subspace S of V, such that $\Omega = G.\pi(\Omega)$ and $\pi(\Omega) = S \cap \Omega$. In [7] the regular convex bodies are classified by showing that they all embed in isotropy representations $K \curvearrowright \mathfrak{p}$ of irreducible noncompact symmetric spaces G/K (where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g} = \mathfrak{lie}(G)$, and K is connected), with $\pi(\Omega) = \mathfrak{a} \cap \Omega$, for $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$.[2] $\pi(\Omega)$ is the convex hull of an orbit (i.e. an *orbitope*) of the action of the Weyl group $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ on $\mathfrak{a}$. Conversely for every regular polytope P occuring as a Weyl orbitope in a maximal abelian subspace in such a symmetric space isotropy representation of a connected compact group K, $K.P$ is a regular convex compact set, uniquely determined up to affine isomorphism by the symmetric space and the (affine isomorphism class of the) polytope P. Listing these symmetric spaces along with the regular polytopes that can occur in them as Weyl orbitopes, which is done in Tables 2, 3, and 4 of [7], classifies the regular convex bodies.

Our proof uses the structure theory from [1] to show that strongly symmetric spectral convex bodies are regular, and uses this theory along with the theory of [8] to show that their Farran-Robertson polytopes are simplices. Focusing on the entries in the tables from [7] for which the Farran-Robertson polytope is a simplex, we verify that these symmetric space representations are either the traceless subspace (the space orthogonal to the Jordan unit) of Jordan algebras, and the associated regular convex bodies affinely

[1] For general probabilistic theories, a reasonable generalization of the space of observables is the vector space V^*.

[2] The representation $K \curvearrowright \mathfrak{p}$ is the restriction to K of the corestriction to $\mathfrak{p}$ of the adjoint representation $G \curvearrowright \mathfrak{g}$.

isomorphic (by translation by a factor of the Jordan unit, into the unit-trace affine plane) to the normalized state space, or, in the few cases that the representations are not manifestly Jordan algebraic, the convex sets are balls, which we can show directly are strongly symmetric and spectral using an argument of Dakić and Brukner [9].

Important aspects of quantum and classical thermodynamics and of query complexity have been generalized to classes of GPTs satisfying natural postulates including or implying spectrality and strong symmetry; our result shows that these apply to a narrower class of theories than might have been hoped, already close to complex quantum theory since their state spaces are Jordan algebraic.

In [10] it was shown that five GPT principles, of which the fifth is strong symmetry, allow the formulation of a reasonable query model generalizing the quantum one, and imply that to have probability $1/2$ or greater of correctly identifying the marked state in Grover's search problem, the number of queries must be at least $(3/2-\sqrt{2})\sqrt{N/k}$, where k is the maximal order of interference of the GPT theory. A lower bound of $\Omega(\sqrt{N})$ was established in the quantum case in [11]; it is achieved by Grover's algorithm. The GPT bound in [10] is also $\Omega(\sqrt{N})$. This limits the potential gain from higher-order interference of degree k in this setting to at most a constant factor, $c/\sqrt{k}$ compared to quantum. However it can be shown (cf. [12]) that the conjunction of the principles used in [10] implies spectrality. Together with Theorem 1 this implies that the GPT systems considered in [10] are Jordan-algebraic, and hence [13, 14] that higher-order interference ($k>2$) is not possible in this setting. In [15], a definition of query computation was formulated and two results were obtained concerning query computation in GPTs under nearly the same assumptions as in [10]; Theorem 1 limits these results, too, to simple Jordan algebraic systems and classical systems.

In [16], it was shown, assuming spectrality and strong symmetry, that the outcome probabilities of any fine-grained measurement on σ are majorized by those of the spectral measurement (which are equal to σ's spectrum), and hence that the measurement entropy determined by any Schur-concave function is equal to the corresponding spectral entropy.[3] In parallel work in [17] the same conclusion was obtained using causality, purification, purity preservation under both parallel and sequential composition of pure operations, and strong symmetry. The first four of these assumptions together imply spectrality. So in light of the present paper, the setting of [16, 17] is no more general than that of simple Jordan-algebraic state spaces, and classical ones.[4] The same conclusions were also obtained from a somewhat different set of premises, defining what are called *sharp theories with purification*, in [12] and [18]. However in [19] it was shown that systems in such theories are also Jordan-algebraic (although the class of systems is not precisely simple Jordan algebras and classical theory, since some nonclassical nonsimple state spaces are definitely allowed, and to the best of our knowledge it is not known whether all simple Jordan algebras are). In [20] the same conclusions were obtained from projectivity of the state space and symmetry of transition probabilities (equivalently, projectivity and self-duality of the state cone, which are in turn equivalent ([21], cf. also [20]) to its perfection together with the normalization of the orthogonal projections onto the linear spans of faces). All Jordan algebraic state spaces have these properties, but it is an open question whether they are the only ones.

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[3]The theorem was stated for the Renyi α-entropies, but the proof uses Schur concavity and applies to arbitrary Schur concave functions.

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