Effectus of Quantum Probability on Relational Structures

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Most of the work presented in this document can be read as a sequel to previous work of the author and collaborators, which has been published and appears in [1]. There, the mathematical description of quantum homomorphisms of graphs and more generally of relational structures, using the language of category theory is given. Specifically, we introduce the concept of 'quantum' monad. The main contribution in the present document consists of providing evidence in support of the claim that the quantum monad fits nicely into the categorical framework of *effectus* theory, developed by Jacobs et al. [15, 6]. We show that the Kleisli category of the quantum monad is an effectus. Also we describe certain aspects of effect-theoretic reasoning like states and predicates, validity and conditioning, instantiated in this effectus.

1 Introduction

Over the past decades, starting with the work of Moggi [28, 29], monads have been widely used to provide a categorical semantics of computation. In this framework, data types are identified with objects $A, B, \dots$ of a given category $\mathbf{C}$, and programs of type B taking parameters of type A with morphisms $A \to TB$ in $\mathbf{C}$. The unary type-constructor T is interpreted as the functor part of a monad defined on $\mathbf{C}$, whose monadic structure allows programs to be composed and thus form a category indeed (the so-called Kleisli category $\mathscr{K}\ell(T)$). By changing the choice of T one obtains different notions of computation such as partiality, non-determinism, side-effects, exceptions, inputs and outputs, etc. For a historical overview of monads within mathematics and logic in computer science, we refer the interested reader to the monograph by Manes [26]. In the present document, we explore several aspects of probability and quantum computation by defining suitable monads on distinct concrete categories (*e.g.* on sets, graphs, or more generally, relational structures). Morphisms in the corresponding Kleisli categories of these monads represent some kind of probabilistic or quantum processes between objects of the underlying categories. Specifically, we shall be interested in the categorical/logical structure of these Kleisli morphisms. This view naturally yields a modular way to reason about the logic of probability and computation, from classical to quantum.

Building on the work of Giry [10], and inspired by algebraic methods in program semantics [23, 22, 8, 33, 30, 32], the study of various 'probability' monads has evolved and became part of a new branch of categorical logic called effectus theory. The main goal of effectus theory is to describe the essentials of quantum computation and logic using the language of category theory. This description includes probabilistic and classical logic and computation as special cases. Various publications have promoted the theory, see *e.g.* [12, 13, 14, 9, 15, 6, 16, 20, 4, 5, 1, 18, 19, 21, 17]. The present work is a contribution to this subject: it shows that the *quantum* monad of [1] fits nicely into this effect-theoretic framework.

An effectus is a rather convenient environment for reasoning about the logic of probability and quantum computation, categorically. More concretely, it is a category with finite coproducts $(+, 0)$ and a final object 1 satisfying two pullback conditions, and one joint monicity requirement (see Definition 2.4). The central feature of an effectus $\mathbf{B}$ is that its maps of type $A \to 1+1$ form an effect module, *i.e.* $\mathbf{B}(A, 1+1) \in \mathbf{EMod}$ for any $A \in \mathbf{B}$; these maps are called predicates and they are thought as the logical abstraction of characteristic functions. This approach emphasises a more 'quantitative' interpretation of predicates as characteristic maps, suitable for quantum and probabilistic *reasoning*. The logic of this predicates is called effect logic [14]. The word 'effect' suggests emphasis in the observer's effect after a measurement procedure has been performed. In quantum computation, one can make use of these effects to show advantages in tasks which require processing information efficiently, such as constraint satisfaction problems via non-local games. See *e.g.* [3, 31, 24, 7, 25, 1, 2]. To put things into context, we start in Section 2 reviewing standard framework in effectus theory, describing the mathematical structures of effect algebras, effect modules, and effectuses in general.

Submitted to:
16th International Conference on Quantum Physics and Logic (QPL 2019)

The main original work presented, in Section 3 of this document, consists of introducing the quantum monad $\mathscr{Q}_d$ on relational structures (see Definition 3.6), which we claim fits into the framework of 'probabilistic' effectuses. We summarise and extend previous work done and published in [1]. Specifically, we show that the Kleisli category of the quantum monad is an effectus (see Theorem 3.1). We also describe the notions of states and predicates, validity and conditioning in $\mathscr{K}\ell(\mathscr{Q}_d)$.

2 Preliminaries

The following presentation is standard in the literature of effectus theory. We shall review only the parts that are needed in our exposition about the Kleisli category of the quantum monad, later in Section 3. For a more detailed account, see for instance [13, 14, 15, 6].

The concept of effect algebras is build on top of the concept of a partial commutative monoid. The prime example of a partial commutative monoid is the real unit interval $[0,1]$.

Definition 2.1. A *partial commutative monoid* consists of a set E with a distinguished element $0 \in E$, and a partial function $\varovee : E \times E \rightharpoonup E$ for which $x \perp y$ denotes $x \varovee y$ is defined, satisfying:

(1) $x \perp y \quad \Rightarrow \quad y \perp x \wedge x \varovee y = y \varovee x$

(2) $y \perp z \wedge x \perp (y \varovee z) \quad \Rightarrow \quad x \perp y \wedge (x \varovee y) \perp z \wedge x \varovee (y \varovee z) = (x \varovee y) \varovee z$

(3) $0 \perp x \wedge 0 \varovee x = x$

The notation $x \perp y$ for saying $x \varovee y \in E$ is defined can also be read as x and y are 'orthogonal' or 'independent'. Axiom (1) is commutativity of $\varovee$, (2) associativity, and (3) zero element. The partial commutative operation $\varovee$ on $[0,1]$ is given by addition $x \varovee y := x + y$ defined only when the sum $x + y$ is less or equal than 1. So, in this case $x \perp y$ denotes $x + y \leq 1$.

Definition 2.2. An *effect algebra* is a partial commutative monoid $(E, 0, \varovee)$ with a unary operation $(-)^{\perp} : E \to E$ satisfying the following axioms:

(1) $\exists! x^{\perp} \in E$ such that $x \varovee x^{\perp} = 1$, where $1 := 0^{\perp}$

(2) $x \perp 1 \quad \Rightarrow \quad x = 0$

A morphism $E \to D$ of effect algebras is defined as a function $f : E \to D$ satisfying the following axioms:

(1) $f(1) = 1$

(2) $x \perp y \quad \Rightarrow \quad f(x) \perp f(y) \wedge f(x \varovee y) = f(x) \varovee f(y)$

Effect algebras form a category denoted by **EA**. Indeed, the real unit interval $[0,1] \in \mathbf{EA}$ with $x^{\perp} := 1 - x$ and $x \varovee y := x + y$ if $x + y \leq 1$ is an effect algebra. Monoids in the category of effect algebras are called effect monoids. (Monoids in the category of commutative rings are called semirings.) Effect monoids form a category denoted by $\mathbf{Mon}(\mathbf{EA})$. The usual multiplication of real numbers turns the unit interval $[0,1] \in \mathbf{Mon}(\mathbf{EA})$ into an effect monoid.

Definition 2.3. An *effect module* is an effect algebra $E \in \mathbf{EA}$ along with a function $\alpha : M \times E \to E$, for some effect monoid $M \in \mathbf{Mon}(\mathbf{EA})$, satisfying the following axioms:

(1) $\alpha(1,1) = 1$

(2) $\alpha(r, -) : M \to M$ and $\alpha(-, x) : M \to E$ are morphisms in **EA**

The action of α can be thought as a scalar multiplication. A map of effect modules is a map of the underlying effect algebras that commutes with scalar multiplication. There is a category $\mathbf{EMod}_M$ of effect modules over M, for any effect monoid $M \in \mathbf{Mon}(\mathbf{EA})$. Fuzzy predicates $X \to [0,1]$ on a set X form an effect module, *i.e.* $[0,1]^X \in \mathbf{EMod}_{[0,1]}$.

Effectus theory is an emergent field in categorical logic aiming to describe logic and probability, from the point of view of classical and quantum computation. The category **Set** of sets and functions has finite coproducts $(+, 0)$ and a terminal object $1 \in \mathbf{Set}$. The disjoint union $+$ of $X, Y \in \mathbf{Set}$ is the set defined as $X + Y := \{(x,0) \mid x \in X\} \cup \{(1,y) \mid y \in Y\}$ with coprojections $X \xrightarrow{\kappa_1} X + Y \xleftarrow{\kappa_2} Y$, and cotupling $[p,q] : X + Y \to Z$ for any pair of maps $X \xrightarrow{p} Z \xleftarrow{q} Y$ given by:

$$[p,q](v) := \begin{cases} p(v) & \text{if } v \in X \\ q(v) & \text{if } v \in Y \end{cases}$$

for all $v \in X + Y$. The empty set $0 := \emptyset \in \mathbf{Set}$ is the initial object, and any choice of a singleton set $1 := \{*\}$ is terminal in **Set**. The unique function $!_X : X \to 1$ is given by $x \mapsto *$ for each $x \in X$. Given functions $f : A \to B$ and $g : X \to Y$ then $f + g : A + X \to B + Y$ is defined as $f + g := [\kappa_1 \circ f, \kappa_2 \circ g]$.

Definition 2.4. An *effectus* **B** is category with finite coproducts $(+,0)$ and a terminal object $1 \in \mathbf{B}$, where the following commutative squares are pullbacks:

$$\begin{array}{ccc} X+Y & \xrightarrow{!_X+\mathrm{id}_Y} & 1+Y \\ {\scriptstyle \mathrm{id}_X+!_Y}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_1+!_Y} \\ X+1 & \xrightarrow[!_X+\mathrm{id}_1]{} & 1+1 \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{!_X} & 1 \\ {\scriptstyle \kappa_1}\downarrow & & \downarrow{\scriptstyle \kappa_1} \\ X+Y & \xrightarrow[!_X+!_Y]{} & 1+1 \end{array}$$

for all $X,Y \in \mathbf{B}$ and the following maps in **B** are jointly monic:

$$(1+1)+1 \underset{\succ_2:=[[\kappa_2,\kappa_1],\kappa_2]}{\overset{\succ_1:=[[\kappa_1,\kappa_2],\kappa_2]}{\rightrightarrows}} 1+1$$

Joint monicity of $\succ_1, \succ_2$ means that given f,g functions: $\succ_1 \circ f = \succ_1 \circ g$ and $\succ_2 \circ f = \succ_2 \circ g$ implies $f = g$.

The category **Set** is the effectus used for modelling classical (deterministic, Boolean) computations. More exactly, we have the following result.

Proposition 2.1. *The category* **Set** *is an effectus.*

Proof. We know how pullbacks are constructed in **Set**. For the first pullback condition from Definition 2.4, let P be the set of pairs $(x,y) \in (X+1)\times(1+Y)$ such that $(!_X+\mathrm{id}_1)(x) = (\mathrm{id}_1+!_Y)(y)$. Note that we have:

$$(X+1)\times(1+Y) \cong (X\times 1)+(1\times 1)+(X\times Y)+(1\times Y)$$

Let $X+1 = X+\{1\}$ and $1+Y = \{0\}+Y$. By cases:

(1) $(x,y) \in X\times\{0\}$ implies $(x,y) = (x,0)$, and so $(!_X+\mathrm{id}_1)(x) = 0 = (\mathrm{id}_1+!_Y)(0)$ for all $x \in X$, thus $X\times 1 \subseteq P$;

(2) $(x,y) \in \{1\}\times\{0\}$ implies $(!_X+\mathrm{id}_1)(1) = 1 \neq 0 = (\mathrm{id}_1+!_Y)(0)$, so $1\times 1 \not\subseteq P$;

(3) $(x,y) \in X\times Y$ implies $(!_X+\mathrm{id}_1)(x) \neq (\mathrm{id}_1+!_Y)(y)$, so $X\times Y \not\subseteq P$;

(4) $(x,y) \in \{1\}\times Y$ implies $(x,y) = (1,y)$, and so $(!_X+\mathrm{id}_1)(1) = 1 = (\mathrm{id}_1+!_Y)(y)$ for all $y \in Y$, thus $1\times Y \subseteq P$.

Hence, the pullback is indeed given by $(X\times 1)+(1\times Y) \cong X+Y$.

For the second pullback condition from Definition 2.4, take $1 = \{0\}$ and consider the set of pairs $(w,0) \in (X+Y)\times 1$ such that $(!_X+!_Y)(w) = \kappa_1(0)$. Note that $(X+Y)\times 1 \cong (X\times 1)+(Y\times 1)$. By cases:

(1) if $(w,0) \in X\times 1$ then $(!_X+!_Y)(w) = 0 = \kappa_1(0)$ for all $w \in X$;

(2) if $(w,0) \in Y\times 1$ then $(!_X+!_Y)(w) = 1 \neq 0 = \kappa_1(0)$ for all $w \in Y$.

Thus the pullback is indeed given by $X\times 1 \cong X$. For the joint monicity requirement from Definitions 2.4, we consider sets $1+1+1 \cong \{a,b,c\}$ and $1+1 \cong \{0,1\}$, and functions $\succ_1, \succ_2 \colon 3 \rightrightarrows 2$ defined as:

$$\succ_1(a) = 0 \qquad \succ_1(b) = \succ_1(c) = 1 \qquad\qquad \succ_2(a) = \succ_2(c) = 1 \qquad \succ_2(b) = 0$$

Further assume we have functions $f,g\colon X \rightrightarrows 3$ such that:

$$\succ_1 \circ f = \succ_1 \circ g \qquad\qquad \succ_2 \circ f = \succ_2 \circ g$$

We need to show that $f = g$. Suppose that $f \neq g$. Then $f(x) \neq g(x)$ for some $x \in X$. Assuming the existence of such x, we arrive to the following contradictions:

- $f(x) = a \Rightarrow g(x) \in \{b,c\} \Rightarrow \succ_1(f(x)) \neq \succ_1(g(x))$
- $f(x) = b \Rightarrow g(x) \in \{a,c\} \Rightarrow \succ_2(f(x)) \neq \succ_2(g(x))$
- $f(x) = c \Rightarrow g(x) \in \{a,b\} \Rightarrow \succ_1(f(x)) \neq \succ_1(g(x))$ if $g(x) = a$, or $\succ_2(f(x)) \neq \succ_2(g(x))$ if $g(x) = b$

Hence it must be the case that $f = g$, and so $\succ_1$ and $\succ_2$ are jointly monic. □

Effectuses are intended to serve also as categorical models for probabilistic (and quantum) logic and computation. To talk about discrete probabilities (categorically) one uses the discrete distribution monad $\mathcal{D}_M$ defined on **Set**, for any effect monoid $M \in \mathbf{Mon}(\mathbf{EA})$ as follows. For any set X, the set $\mathcal{D}_M(X)$ consists of all finite convex combinations of elements from X with 'mixing probabilities' from M, *i.e.* finite formal sums $m_1|x_1\rangle + \cdots + m_r|x_r\rangle \in \mathcal{D}_M(X)$ where $x_i \in X$, $m_i \in M$ and $m_1 \ovee \cdots \ovee m_r = 1$. An element $\omega \in \mathcal{D}_M(X)$ is called a state/distribution on X, and can also be thought as a function $\omega\colon X \to M$ with finite and orthogonal support $\mathrm{supp}(\omega) := \{x \in X \mid \omega(x) \neq 0\}$, satisfying $\bigovee_{x\in X} \omega(x) = 1$. If $\mathrm{supp}(\omega) = \{x_1,\ldots,x_r\}$ then the assignment $\omega(x_i) \mapsto m_i$ gives the bijective correspondence between these two equivalent representations (*i.e.* as convex-sums, or as mixing-functions) of states. States on X can be coarse-grained along a map $f\colon X \to Y$ to get states on Y: for any state $\omega \in \mathscr{D}_M(X)$ there is a state $\mathscr{D}_M(f)(\omega) \in \mathscr{D}_M(Y)$, given by $\mathscr{D}_M(f)(\omega)(y) := \bigovee_{x \in f^{-1}(y)} \omega(x)$. Now we describe the monadic structure of $\mathscr{D}_M$. The unit $\eta_X : X \to \mathscr{D}_M(X)$ is the Dirac delta distribution, *i.e.* $\eta_X(x)(x') = 1$ if $x = x'$ and $\eta_X(x)(x') = 0$ if $x \neq x'$. The multiplication $\mu_X : \mathscr{D}_M^2(X) \to \mathscr{D}_M(X)$ is given by the expectation-value of evaluation functions $\omega \mapsto \omega(x)$ with respect to some distribution of distributions $\Omega \in \mathscr{D}_M^2(X)$, *i.e.* $\mu_X(\Omega)(x) := \bigovee_{\omega \in \mathscr{D}_M(X)} \Omega(\omega) \cdot \omega(x)$. Therefore, there is a (M-valued discrete) distribution monad $\mathscr{D}_M = (\mathscr{D}_M, \eta, \mu)$ on **Set** for each effect monoid $M \in \mathbf{Mon}(\mathbf{EA})$.

The Kleisli category $\mathscr{K}\ell(\mathscr{D}_M)$ of the distribution monad $\mathscr{D}_M$ on **Set** has sets as objects, and morphisms $X \to Y$ in $\mathscr{K}\ell(\mathscr{D}_M)$ are precisely functions of type $X \to \mathscr{D}_M(Y)$ in **Set**. For every $X \in \mathbf{Set}$, the identity map $X \to X$ in $\mathscr{K}\ell(\mathscr{D}_M)$ is given by the unit $\eta_X \colon X \to \mathscr{D}_M(X)$. One can define the Kleisli extension $c_* \colon \mathscr{D}_M(X) \to \mathscr{D}_M(Y)$ of a Kleisli map $c\colon X \to \mathscr{D}_M(Y)$ as $c_* := \mu_Y \circ \mathscr{D}_M(c)$. More concretely, this is $c_*(\omega)(y) = \bigovee_{x \in X} \omega(x) \cdot c(x)(y)$ for all $\omega \in \mathscr{D}_M(X)$ and $y \in Y$. Composition of Kleisli maps $c\colon X \to \mathscr{D}_M(Y)$ and $d\colon Y \to \mathscr{D}_M(Z)$, is given using Klesli extension (to simplify notation) as:

$$X \xrightarrow[= \mu_Z \circ \mathscr{D}_M(d) \circ c]{d_* \circ\, c} \mathscr{D}_M(Z)$$

Remark 2.1. For any set function $f\colon X \to Y$, one can define a Kleisli map:

$$X \xrightarrow[= \mathscr{D}_M(f) \circ \eta_X]{\hat{f} := \eta_Y \circ f} \mathscr{D}_M(Y)$$

given by naturality of η.

Thus, the category $\mathscr{K}\ell(\mathscr{D}_M)$ has finite coproducts $(0, +)$ given by the empty set $0 := \emptyset \in \mathbf{Set}$, and disjoint union $X_1 + X_2$ with coprojections:

$$X_i \xrightarrow[= \mathscr{D}_M(\kappa_i) \circ \eta_{X_i}]{\hat{\kappa}_i := \eta_{(X_1+X_2)} \circ \kappa_i} \mathscr{D}_M(X_1 + X_2)$$

for $i = 1, 2$. The following result is well-known, see *e.g.* [13, Proposition 6.4].

Lemma 2.1. *The distribution monad $\mathscr{D}_M$ is affine, i.e. $\mathscr{D}_M(1) \cong 1$, for any effect monoid $M \in \mathbf{Mon}(\mathbf{EA})$. Moreover, $\mathscr{D}_M(2) \cong M$.*

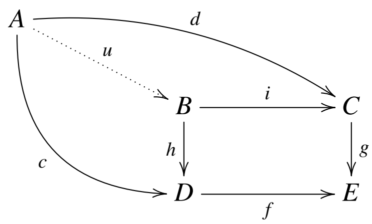
Proof. An element $\omega \in \mathscr{D}_M(1)$ can be regarded as a function $\omega \colon 1 \to M$ such that $\bigovee_{x \in \{1\}} \omega(x) = 1$. Therefore, it must be the case that $\omega(1) = 1$. Thus $\mathscr{D}_M(1) \cong 1$. Now, an M-valued distribution over a 2-element set consists of a choice of an element $m \in M$, which in turn immediately determines a choice of $m^\perp \in M$ such that $m \ovee m^\perp = 1 \in M$. Hence $\mathscr{D}_M(2) \cong M$. □

By Lemma 2.1 above, any choice of a singleton set $1 \in \mathbf{Set}$ is a terminal object in $\mathscr{K}\ell(\mathscr{D}_M)$ so we have unique arrows:

$$X \xrightarrow[= \mathscr{D}_M(!_X) \circ \eta_X]{\hat{!}_X := \eta_1 \circ !_X} \mathscr{D}_M(1)$$

for any $X \in \mathbf{Set}$. Since $1 \cong \mathscr{D}_M(1)$, the unit $\eta_1 \colon 1 \to \mathscr{D}_M(1)$ and the identity function $\mathrm{id}_1 \colon 1 \to 1$ are equal $\eta_1 = \mathrm{id}_1$. Therefore, we have $\hat{!}_X = !_X$ for all $X \in \mathbf{Set}$. So $\mathscr{K}\ell(\mathscr{D}_M)$ has finite coproducts and a terminal object. Thus, we would like to have the two pullbacks from Definition 2.4 instantiated in $\mathscr{K}\ell(\mathscr{D}_M)$. Let's describe this situation in general first.

Remark 2.2. Assume we have the following commutative diagram:



where all the arrows live in $\mathscr{K}\ell(\mathscr{D}_M)$, and the dashed arrow means that u is uniquely defined. That is, we have a function $u\colon A \to \mathscr{D}_M(B)$ which is determined in a unique way given Kleisli maps:

$$\begin{array}{lll} c\colon A \to \mathscr{D}_M(D) & f\colon D \to \mathscr{D}_M(E) & h\colon B \to \mathscr{D}_M(D) \\ d\colon A \to \mathscr{D}_M(C) & g\colon C \to \mathscr{D}_M(E) & i\colon B \to \mathscr{D}_M(C) \end{array}$$

satisfying the following four equations:

$$h_* \circ u = c \tag{1}$$
$$i_* \circ u = d \tag{2}$$
$$f_* \circ c = g_* \circ d \tag{3}$$
$$f_* \circ h = g_* \circ i \tag{4}$$

In that case, we have that B is the pullback of g along f in $\mathscr{K}\ell(\mathscr{D}_M)$.

Proposition 2.2. *Let $M \in \mathbf{Mon}(\mathbf{EA})$ be an effect monoid. The Kleisli category $\mathscr{K}\ell(\mathscr{D}_M)$ of the distribution monad $\mathscr{D}_M$ on sets is an effectus.*

Proof. We need to check two pullback conditions and one joint monicity requirement for the Kleisli category $\mathscr{K}\ell(\mathscr{D}_M)$. We start with the first pullback from Definition 2.4. We assume to have the following Kleisli maps:

$$\begin{array}{lll} c\colon A \to \mathscr{D}_M(X+1) & f\colon X+1 \to \mathscr{D}_M(1+1) & h\colon X+Y \to \mathscr{D}_M(X+1) \\ d\colon A \to \mathscr{D}_M(1+Y) & g\colon 1+Y \to \mathscr{D}_M(1+1) & i\colon X+Y \to \mathscr{D}_M(1+Y) \end{array}$$

where:

$$\begin{array}{ll} f := \mathscr{D}_M(!_X + \mathrm{id}_1) \circ \eta_{X+1} & h := \mathscr{D}_M(\mathrm{id}_X + !_Y) \circ \eta_{X+Y} \\ g := \mathscr{D}_M(\mathrm{id}_1 + !_Y) \circ \eta_{1+Y} & i := \mathscr{D}_M(!_X + \mathrm{id}_Y) \circ \eta_{X+Y} \end{array}$$

By definition of Kleisli extension we have:

$$\begin{aligned} f_* &= \mu_{1+1} \circ \mathscr{D}_M(f) \\ &= \mu_{1+1} \circ \mathscr{D}_M(\mathscr{D}_M(!_X + \mathrm{id}_1) \circ \eta_{X+1}) \\ &\stackrel{\star}{=} \mu_{1+1} \circ \mathscr{D}_M(\eta_{1+1} \circ (!_X + \mathrm{id}_1)) \\ &= \mu_{1+1} \circ \mathscr{D}_M(\eta_{1+1}) \circ \mathscr{D}_M(!_X + \mathrm{id}_1) \\ &= \mathscr{D}_M(!_X + \mathrm{id}_1) \end{aligned}$$

where the marked equality $\stackrel{\star}{=}$ follows from naturality of η, and the last one from the axioms of monads. Similarly, we have:

$$\begin{aligned} g_* &= \mathscr{D}_M(\mathrm{id}_1 + !_Y) \\ h_* &= \mathscr{D}_M(\mathrm{id}_X + !_Y) \\ i_* &= \mathscr{D}_M(!_X + \mathrm{id}_Y) \end{aligned}$$

Therefore, equation (4) above holds:

$$\begin{aligned} f_* \circ h &= \mathscr{D}_M(!_X + \mathrm{id}_1) \circ h \\ &= \mathscr{D}_M(!_X + \mathrm{id}_1) \circ \mathscr{D}_M(\mathrm{id}_X + !_Y) \circ \eta_{X+Y} \\ &= \mathscr{D}_M((!_X + \mathrm{id}_1) \circ (\mathrm{id}_X + !_Y)) \circ \eta_{X+Y} \\ &\stackrel{\star}{=} \mathscr{D}_M((\mathrm{id}_1 + !_Y) \circ (!_X + \mathrm{id}_Y)) \circ \eta_{X+Y} \\ &= \mathscr{D}_M(\mathrm{id}_1 + !_Y) \circ \mathscr{D}_M(!_X + \mathrm{id}_Y) \circ \eta_{X+Y} \\ &= g_* \circ i \end{aligned}$$

where the marked equality $\stackrel{\star}{=}$ follows from the fact that both squares in the definition of effectus (see Definition 2.4) commute in every category with finite coproducts and a terminal object.

Let $X+1 = X + \{1\}$ and $1+Y = \{0\}+Y$. Further suppose the Kleisli maps $c\colon A \to \mathscr{D}_M(X+\{1\})$ and $d\colon A \to \mathscr{D}_M(\{0\}+Y)$ satisfy equation (3) above. More concretely, suppose:

$$\mathscr{D}_M(!_X + \mathrm{id}_Y)(c(a)) = \mathscr{D}_M(\mathrm{id}_X + !_Y)(d(a)) \quad \in \quad \mathscr{D}_M(\{0\}+\{1\}) \tag{5}$$

for all $a \in A$. Specifically, this equation (5) expanded and evaluated says that:

$$\begin{aligned} \mathscr{D}_M(!_X + \mathrm{id}_Y)(c(a))(0) &\stackrel{(5)}{=} \mathscr{D}_M(\mathrm{id}_X + !_Y)(d(a))(0) \\ &= \bigovee_{y \in (\mathrm{id}_X + !_Y)^{-1}(0)} d(a)(y) \\ &= d(a)(0) \quad \in \quad M \end{aligned} \tag{6}$$

$$\begin{aligned}\mathscr{D}_M(!_X+\mathrm{id}_Y)(c(a))(1) &= \bigovee_{x\in(!_X+\mathrm{id}_Y)^{-1}(1)} c(a)(x) \\ &= c(a)(1) \quad \in \quad M\end{aligned} \tag{7}$$

Thus:

$$d(a)(0)\ovee c(a)(1) = 1 \in M \tag{8}$$

Let $u\colon A\to\mathscr{D}_M(X+Y)$ be the Kleisli map defined as $u(a)(x) := c(a)(x)\in M$ for all $x\in X$ and $u(a)(y) := d(a)(y)\in M$ for all $y\in Y$. We have:

$$\begin{aligned}\bigovee_{x\in X} u(a)(x)\ovee\bigovee_{y\in Y} u(a)(y) &\overset{\text{def}}{=} \bigovee_{x\in X} c(a)(x)\ovee\bigovee_{y\in Y} d(a)(y) \\ &= \bigovee_{!_X(x)=0} c(a)(x)\ovee\bigovee_{!_Y(y)=1} d(a)(y) \\ &= \mathscr{D}_M(!_X+\mathrm{id}_Y)(c(a))(0)\ovee\mathscr{D}_M(!_X+\mathrm{id}_Y)(c(a))(1) \\ &\overset{\star}{=} d(a)(0)\ovee c(a)(1) \\ &= 1\end{aligned}$$

where the marked equality $\overset{\star}{=}$ follows from (6) and (7), and the last equality from (8). Hence u is well-defined. We still need to check (1) and (2) above, which in this case amounts to show that:

$$\begin{aligned}\mathscr{D}_M(\mathrm{id}_X+!_Y)\circ u &= c \\ \mathscr{D}_M(!_X+\mathrm{id}_Y)\circ u &= d\end{aligned}$$

For all $a\in A$, we have indeed:

$$\begin{aligned}\mathscr{D}_M(\mathrm{id}_X+!_Y)(u(a))(x) &= \bigovee_{x'\in(\mathrm{id}_X+!_Y)^{-1}(x)} u(a)(x') \\ &= u(a)(x) \\ &\overset{\text{def}}{=} c(a)(x) \\ \mathscr{D}_M(\mathrm{id}_X+!_Y)(u(a))(1) &= \bigovee_{y\in(\mathrm{id}_X+!_Y)^{-1}(1)} u(a)(y) \\ &= \bigovee_{y\in Y} u(a)(y) \\ &\overset{\text{def}}{=} \bigovee_{y\in Y} d(a)(y) \\ &= c(a)(1) \\ \mathscr{D}_M(!_X+\mathrm{id}_Y)(u(a))(y) &= \bigovee_{y'\in(!_X+\mathrm{id}_Y)^{-1}(y)} u(a)(y') \\ &= u(a)(y) \\ &\overset{\text{def}}{=} d(a)(y) \\ \mathscr{D}_M(!_X+\mathrm{id}_Y)(u(a))(0) &= \bigovee_{x\in(!_X+\mathrm{id}_Y)^{-1}(0)} u(a)(x) \\ &= \bigovee_{x\in X} u(a)(x) \\ &\overset{\text{def}}{=} \bigovee_{x\in X} c(a)(x) \\ &= d(a)(0).\end{aligned}$$

By definition, $u\colon A\to\mathscr{D}_M(X+Y)$ is the unique Kleisli map satisfying the needed requirements. This completes the proof of the first pullback condition for $\mathscr{K}\ell(\mathscr{D}_M)$.

For the second pullback from Definition 2.4, let $1 := \{0\}$ and $1 := \{1\}$ be two distinct (choices of) singleton sets, and $\theta = 1|0\rangle\in\mathscr{D}_M(1)$ where $\mathscr{D}_M(1)\cong\{\theta\}$ and $1\in M$ is a (Dirac) distribution on the first singleton $\{0\}$ defined above. Consider Kleisli maps $!_A\colon A\to\{\theta\}$ and $c\colon A\to\mathscr{D}_M(X+Y)$ such that:

$$\mathscr{D}_M(!_X+!_Y)\circ c = \mathscr{D}_M(\kappa_1)\circ\,!_A \tag{9}$$

Since $c(a) = \sum_x m_x|x\rangle + \sum_y m_y|y\rangle \in \mathscr{D}_M(X+Y)$ with $\bigovee_x m_x \ovee \bigovee_y m_y = 1 \in M$ for all $a \in A$, we have that the left-hand side of equation (9) expands to:

$$\mathscr{D}_M(!_X+!_Y)(c(a)) = \sum_x m_x|0\rangle + \sum_y m_y|1\rangle$$

The right-hand side of equation (9)expands to:

$$\begin{aligned}\mathscr{D}_M(\kappa_1)(!_A(a)) &= \mathscr{D}_M(\kappa_1)(\theta) \\ &= 1|\kappa_1(0)\rangle \\ &= 1|0\rangle\end{aligned}$$

Hence $\bigovee_x m_x = 1$, and so $c(a) \in \mathscr{D}_M(X)$. Let $u\colon A \to \mathscr{D}_M(X)$ be defined as $u(a)(x) := c(a)(x)$. By definition, the Kleisli map $u\colon A \to \mathscr{D}_M(X)$ is the unique arrow satisfying the needed requirements.

Now we prove that the maps $\succ_1, \succ_2\colon (1+1)+1 \rightrightarrows 1+1$ in are jointly monic in $\mathcal{K}\ell(\mathscr{D}_M)$. This part is taken exactly from [15, Example 4.7]. Let $\sigma, \tau \in \mathscr{D}_M(3)$ be distributions such that

$$\begin{aligned}\mathscr{D}_M(\succ_1)(\sigma) &= \mathscr{D}_M(\succ_1)(\tau) \\ \mathscr{D}_M(\succ_2)(\sigma) &= \mathscr{D}_M(\succ_2)(\tau)\end{aligned} \tag{10}$$

in $\mathscr{D}_M(2)$. Assume $3 = \{a,b,c\}$ and $2 = \{0,1\}$. We have the following convex combinations for σ in $\mathscr{D}_M(2)$:

$$\begin{aligned}\mathscr{D}_M(\succ_1)(\sigma) &= \sigma(a)|0\rangle + (\sigma(b)+\sigma(c))|1\rangle \\ \mathscr{D}_M(\succ_2)(\sigma) &= \sigma(b)|0\rangle + (\sigma(a)+\sigma(b))|1\rangle\end{aligned}$$

Similarly for τ:

$$\begin{aligned}\mathscr{D}_M(\succ_1)(\tau) &= \tau(a)|0\rangle + (\tau(b)+\tau(c))|1\rangle \\ \mathscr{D}_M(\succ_2)(\tau) &= \tau(b)|0\rangle + (\tau(a)+\tau(b))|1\rangle\end{aligned}$$

Hence, by the first equation in (10), we have $\sigma(a) = \tau(a)$. Similarly, by the second equation in (10), we have $\sigma(b) = \tau(b)$. We still need to show that $\sigma(c) = \tau(c)$. Since $\sigma(a) \ovee \sigma(b) \ovee \sigma(c) = 1 = \tau(a) \ovee \tau(b) \ovee \tau(c)$, then:

$$\begin{aligned}\sigma(c) &= (\sigma(a) \ovee \sigma(b))^{\perp} \\ &= (\tau(a) \ovee \tau(b))^{\perp} \\ &= \tau(c)\end{aligned}$$

This completes the proof. □

3 The Quantum Monad on Relational Structures

Measurement is a central aspect in any frequentist interpretation of probability. Quantum theory is a physical theory of measurements in the sense that it provides a framework to build models for predicting the probability distributions of observable properties (aka ‘observables’). Physical indeed because the probabilities given by the distributions are interpreted as statistical frequencies of observables, after a measurement procedure has been performed repeatedly for a sufficient number of times. In quantum computation, one uses the mathematical representation of quantum systems and measurements for processing information more efficiently, like finding solutions to systems of polynomial equations for which it is known there are no classical solutions, as for instance in proving existence of *e.g.* non-classical ‘quantum’ perfect strategies for non-local games [3, 31, 24, 7, 25, 1, 2].

3.1 Quantum Graph Homomorphisms

To begin our formal discussion about another ‘probabilistic’ monad, we shall consider graphs as they are rather generic objects.

Definition 3.1. A *graph* G consists of a set of vertices $V(G)$, together with a set of edges $E(G) \subseteq V(G) \times V(G)$ which are pairs of adjacent vertices.

By definition $E(G)$ is a binary relation on the vertex set $V(G)$. We shall write $v \sim v'$ to denote a pair of adjacent vertices $v, v' \in V(G)$, *i.e.* a pair $(v,v') \in V(G) \times V(G)$ in the edge/adjacency relation $(v,v') \in E(G)$. A morphism $G \to H$ of graphs is given by a function $f\colon V(G) \to V(H)$ between vertices preserving edge adjacency: if $v \sim v'$ in G then $f(v) \sim f(v')$ in H. Morphisms of graphs are also known as *graph homomorphisms*. Graphs and their homomorphisms form a category denoted by **Gph**.

Remark 3.1. Essentially, the category **Gph** defined as above is the category **Rel** of binary relations on sets: there is a category **Rel** whose objects are pairs (A,R) where $R \subseteq A \times A$ is a (binary) relation on $A \in \mathbf{Set}$, and morphisms $(A,R) \to (B,S)$ are functions $f\colon A \to B$ between the underlying sets such that $(a,a') \in R$ implies $(f(a), f(a')) \in S$ for all $a, a' \in A$ (see [11, Chapter 0]). Hence, we have $\mathbf{Gph} \cong \mathbf{Rel}$ by definition. Recall that $\mathbf{Sub}(\mathbf{Set})$ is the category with pairs (A,X) where $X \subseteq A$ is subset of a set $A \in \mathbf{Set}$ as objects, and functions $f\colon A \to B$ satisfying $a \in X$ implies $f(a) \in Y \subseteq B$ for each $a \in A$ as morphisms. The forgetful functor $\mathbf{Gph} \to \mathbf{Set}$ which maps a graph to its vertex set $G \mapsto V(G)$ is a bifibration obtained, by taking the ordinary pullback of categories (*i.e.* pullback in the category **Cat** of small categories and functors), from the subsets fibration $\mathbf{Sub}(\mathbf{Set}) \to \mathbf{Set}$ defined by $(A,X) \mapsto A$, as follows:

$$\begin{array}{ccc} \mathbf{Gph} & \longrightarrow & \mathbf{Sub}(\mathbf{Set}) \\ \big\downarrow \lrcorner & & \big\downarrow \\ \mathbf{Set} & \xrightarrow[A \,\mapsto\, A\times A]{} & \mathbf{Set} \end{array}$$

Actually, more can be said about the bifibration $\mathbf{Gph} \to \mathbf{Set}$. For instance, it preserves finite products and coproducts (see [11, Example 9.2.5 (ii)]).

Remark 3.2. From now on, throughout this document, we use the word graph to refer to *simple*, *irreflexive* and *undirected* graphs always, unless otherwise is explicitly stated. Simple means that there is at most one edge between pairs of vertices. Irreflexive means that $E(G)$ is not a reflexive relation, *i.e.* $v \not\sim v$ for all $v \in V(G)$. Undirected means that $E(G)$ is a symmetric relation, *i.e.* $v \sim v'$ if and only if $v' \sim v$. However, for the sake of convenience, the category **Gph** shall include all type of graphs in the sense of Definition 3.1.

The language of graphs is simple, yet powerful enough to talk about key aspects of logic and computation (see Remark). For instance, consider the following game involving a given pair of graphs G and H, played by Alice and Bob cooperating against a Verifier. Their goal is to establish the existence of a graph homomorphism from G to H. The game is 'non-local' which means that Alice and Bob are not allowed to communicate during the game, however they are allowed to agree on a strategy before the game has started. In each round Verifier sends to Alice and Bob vertices $v_1, v_2 \in V(G)$, respectively; in response they produce outputs $w_1, w_2 \in V(H)$. They win the round if the following conditions hold:

$$v_1 = v_2 \Rightarrow w_1 = w_2 \qquad \text{and} \qquad v_1 \sim v_2 \Rightarrow w_1 \sim w_2$$

If there is indeed a graph homomorphism $G \to H$, then Alice and Bob can win any round of the game described above by using such homomorphism as strategy for responding accordingly. Conversely, they can win any round with certainty only when there is a graph homomorphism $G \to H$. A strategy for Alice and Bob in which they win with probability 1 is called a *perfect strategy*. Hence, the existence of a perfect strategy is equivalent to the existence of a graph homomorphism. Using quantum resources in the form of a maximally entangled bipartite state, where Alice and Bob can each perform measurements on their part, there are perfect strategies in cases where no classical homomorphism exists.

We write $M_d(\mathbb{C})$ for the set of all $d \times d$ matrices with complex entries ($d \in \mathbb{N}$). Also, we write $1 \in M_d(\mathbb{C})$ for the $d \times d$ identity matrix. Let $E \in M_n(\mathbb{C})$ and $F \in M_m(\mathbb{C})$ be two complex square matrices of possibly different size. Their *tensor product* is defined as $E \otimes F := (e_{ij}F) \in M_{nm}(\mathbb{C})$ if $E = (e_{ij})$ with $i, j \in \{1, \ldots, n\}$.

Definition 3.2. A *quantum perfect strategy* for the homomorphism game from G to H consists of a complex unitary vector $\psi \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ for some $d_A, d_B \in \mathbb{N}$ finite, and families $(E_{vw})_{w \in V(H)}$ and $(F_{vw})_{w \in V(H)}$ of $d_A \times d_A$ and $d_B \times d_B$ complex matrices for all $v \in V(G)$, satisfying:

(1) $\sum_{w \in V(H)} E_{vw} = 1 \in M_{d_A}(\mathbb{C})$ and $\sum_{w \in V(H)} F_{vw} = 1 \in M_{d_B}(\mathbb{C})$;

(2) $w \neq w' \quad \Rightarrow \quad \psi^*(E_{xy} \otimes F_{vw'})\psi = 0$;

(3) $v \sim v' \wedge w \not\sim w' \quad \Rightarrow \quad \psi^*(E_{vw} \otimes F_{v'w'})\psi = 0$.

This characterisation of quantum perfect strategies eliminates the two-person aspect of the game and the shared state, leaving a 'matrix-valued relation' as the witness for existence of a quantum perfect strategy. It also gives rise to the notion of 'quantum' graph homomorphism. This concept was introduced in [31], as a generalisation of the notion of quantum chromatic number from [3]. Analogous results for constraint systems are proved in [24, 7, 25, 1, 2].

Definition 3.3. A *quantum graph homomorphism* from G to H is given by an indexed family $(E_{vw})_{v \in V(G), w \in V(H)}$ of $d \times d$ complex matrices ($E_{vw} \in M_d(\mathbb{C})$), for some $d \in \mathbb{N}$, such that:

(1) $E_{vw}^* = E_{vw}^2 = E_{vw}$ for all $v \in V(G)$ and $w \in V(H)$;

(2) $\sum_{w \in V(H)} E_{vw} = 1 \in M_d(\mathbb{C})$ for all $v \in V(G)$;

(3) $(v = v' \wedge w \neq w') \vee (v \sim v' \wedge w \not\sim w') \quad \Rightarrow \quad E_{vw}E_{v'w'} = 0$.

An important further step is taken in [31]: a construction $G \mapsto \mathsf{M}G$ on graphs is introduced, such that the existence of a quantum graph homomorphism from G to H is equivalent to the existence of a graph homomorphism of type $G \to \mathsf{M}H$. This construction is called the *measurement graph*, and it turns out to be a graded monad on the category of graphs. Hence the Kleisli maps of this monad are exactly the 'quantum' maps between graphs of [31, 24, 25, 2].

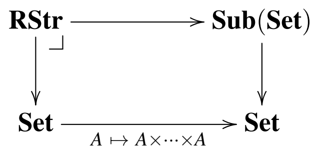
3.2 The Quantum Monad

A simple undirected graph G is a relational structure with a single, binary irreflexive and symmetric relation $E(G)$ that we have been written as $\sim$ in infix notation. The objects of the category **Gph** (as in Definition 3.1) are sets together with a binary relation. Relational structures are even more general.

Definition 3.4. A *relational structure* $\mathscr{A}$ consists of a set A together with an indexed family $R(\mathscr{A}) = (R_i^{\mathscr{A}})_{i\in I}$ of relations $R_i^{\mathscr{A}} \subseteq A^{k_i}$ with $I \in \mathbf{Set}$, and $k_i \in \mathbb{N}$ for all $i \in I$.

A map of relational structures $\mathscr{A} \to \mathscr{B}$ is a function $f\colon A \to B$ between the underlying sets, preserving all relations: $(x_1,\dots,x_k) \in R^{\mathscr{A}} \Rightarrow (f(x_1),\dots,f(x_k)) \in R^{\mathscr{B}}$ for all $(x_1,\dots,x_k) \in A^k$ and all $R \in R(\mathscr{A})$ with arity $k \in \mathbb{N}$. In fact, there is a category of relational structures that we denote by **RStr**. This category **RStr** has a relationship with the category **Rel** of binary relations defined before. By definition, **Rel** $\cong$ **Gph** is a (full) subcategory of **RStr**.

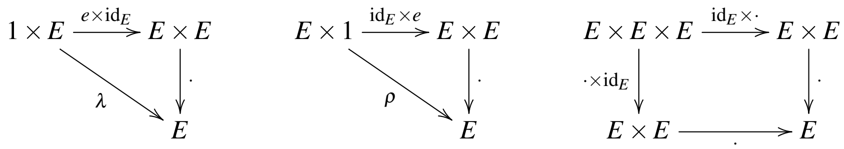
Remark 3.3. For the sake of simplicity (wrt. notation), we shall assume that all the relational structures have only one relation of a fixed arity $k \in \mathbb{N}$, *i.e.* $R(\mathscr{A}) = \{R^{\mathscr{A}}\}$ and $R^{\mathscr{A}} \subseteq A^k$ for all $\mathscr{A} \in \mathbf{RStr}$. That is, the category **RStr** is obtained from the fibration $\mathbf{Sub}(\mathbf{Set}) \to \mathbf{Set}$ of subsets by taking the pullback:



Since $\mathbf{RStr} \to \mathbf{Set}$ is obtained from the above change-of-base situation, it is a bifibration by construction. This is indeed very similar to the case of the category **Gph** (see Remark 3.1). In particular, $\mathbf{RStr} \to \mathbf{Set}$ is defined by $\mathscr{A} \mapsto A$, and it also preserves finite products and coproducts.

There is a category **Pre** of preordered sets with monotone functions. A preorder is a reflexive and transitive binary relation. Preordered sets are sets carrying a preorder.

Definition 3.5. A *preordered monoid* $\mathbb{E} = (E, \sqsubseteq, *, \cdot)$ consists of a preordered set $(E, \sqsubseteq) \in \mathbf{Pre}$ with a distinguished element $* \in E$, together with a monotone function $\cdot\colon (E,\sqsubseteq) \times (E,\sqsubseteq) \to (E,\sqsubseteq)$ satisfying the axioms of monoids:



where $1 = \{*\}$ is a one-element set, and λ and ρ are bijections given by $\lambda(*,e) := e$ and $\rho(e,*) := e$ for all $e \in E$.

Preordered monoids are monoids in the cateogry of preordered sets **Pre**. Preordered monoids form a category **Mon**(**Pre**). Graded monads on an arbitrary category **C** can be defined as preordered monoids in the (functor) category $[\mathbf{C},\mathbf{C}]$ of endofunctors $\mathbf{C} \to \mathbf{C}$ and natural transformations between them. The natural numbers $\mathbb{N}$ with $1 \in \mathbb{N}$ as the distinguished element is a preordered monoid, and is in fact the one we use for grading the quantum monad [1]. The measurement graph construction from [31], which we mentioned in the last paragraph of the previous section, is an instance of the *quantum* monad $\mathscr{Q}_d$ defined on **RStr** instantiated in the category of (simple, undirected) graphs. We write $\mathrm{Proj}(d) \subseteq M_d(\mathbb{C})$ for the set of all $d \times d$ complex matrices that are both self-adjoint and idempotent, *i.e.* $\mathrm{Proj}(d) = \{a \in M_d(\mathbb{C}) \mid a^* = a^2 = a\}$ for all $d \in \mathbb{N}$. The set $\mathrm{Proj}(d)$ of 'projectors' (of dimension $d \in \mathbb{N}$) is an effect monoid with the usual operations of sum and product of matrices, defined partially: $p+q$ is defined only when projectors $p,q \in \mathrm{Proj}(d)$ are orthogonal $p \cdot q = 0$, and $p \cdot q$ only when they commute $p \cdot q = q \cdot p$. That is, for all $d \in \mathbb{N}$ we have $\mathrm{Proj}(d) \in \mathbf{Mon}(\mathbf{EA})$. The functor part of the quantum monad $\mathscr{Q}_d \in \mathbf{Mon}([\mathbf{RStr},\mathbf{RStr}])$ is defined as follows.

Definition 3.6. For every relational structure $\mathscr{A} = (A, R^{\mathscr{A}})$, let $\mathscr{Q}_d(A) := \mathscr{D}_M(A)$ with $M := \mathrm{Proj}(d)$, be the set of *projection-valued* distributions on A, where $\mathscr{D}_M$ is the distribution functor/monad. The k-ary relation $R^{\mathscr{Q}_d(\mathscr{A})} \subseteq \mathscr{Q}_d(A)^k$ is defined as $(p_1,\dots,p_k) \in R^{\mathscr{Q}_d(\mathscr{A})}$ if and only if the projection-valued distributions $p_1,\dots,p_k \in \mathscr{Q}_d(A)$ satisfy the following two conditions: (1) $p_i(x)$ and $p_j(x')$ commute for all $x,x' \in A$, and (2) $(x_1,\dots,x_k) \notin R^{\mathscr{A}}$ implies $\prod_{i=1}^k p_i(x_i) = 0$ for all $(x_1,\dots,x_k) \in A^k$.

Remark 3.4. An element $p \in \mathscr{Q}_d(A)$ is a map $p\colon A \to \mathrm{Proj}(d)$ satisfying $\sum_{x\in A} p(x) = 1$. Note that because of the normalisation condition, all these projectors $p(x)$ resolving the identity 1 are pairwise orthogonal.

This was the definition of the relational structure $\mathscr{Q}_d(\mathscr{A}) = (\mathscr{Q}_d(A), R^{\mathscr{A}}) \in \mathbf{RStr}$. Given a map $f\colon \mathscr{A} \to \mathscr{B}$ of relational structures, we define $\mathscr{Q}_d(f)\colon \mathscr{Q}_d(\mathscr{A}) \to \mathscr{Q}_d(\mathscr{B})$ by coarse-graining $p\colon A \to \mathrm{Proj}(d)$ along f as given by the formula: $\mathscr{Q}_d(f)(p)(y) = \sum_{x\in f^{-1}(y)} p(x)$ for all $y \in B$. This is a well-defined homomorphism between relational structures. This definition preserves composites and identities, so there is a functor $\mathscr{Q}_d\colon \mathbf{RStr} \to \mathbf{RStr}$ for every dimension $d \in \mathbb{N}$ (see [1] for more details).

Note that $\mathrm{Proj}(1) = \{0,1\} \cong 2 = 1+1$. We define $\eta_{\mathscr{A}} \colon \mathscr{A} \to \mathscr{Q}_1(\mathscr{A})$ to be a Dirac delta distribution: $\eta_{\mathscr{A}}(x)(x') = 1$ if $x = x'$ and $\eta_{\mathscr{A}}(x)(x') = 0$ if $x \neq x'$. Verification that this is well-defined is straight-forward. For $\mu^{d,d'}_{\mathscr{A}} \colon \mathscr{Q}_d\mathscr{Q}_{d'}(\mathscr{A}) \to \mathscr{Q}_{dd'}(\mathscr{A})$, let $\mu^{d,d'}_{\mathscr{A}}(P)(x) = \sum_{p\in\mathscr{Q}_{d'}(A)} P(p)\otimes p(x) \in \mathrm{Proj}(dd')$. Verification that this is well-defined is also straight-forward: one just need to recall that all of our distributions have finite support. These two maps turn out to be components of two natural transformations $\eta_{\mathscr{A}} \colon 1 \Rightarrow \mathscr{Q}_1$ and $\mu^{d,d'}_{\mathscr{A}} \colon \mathscr{Q}_d\mathscr{Q}_{d'} \Rightarrow \mathscr{Q}_{dd'}$ satisfying the axioms of graded monads [27]: $\mu^{d,1}_{\mathscr{A}} \circ \mathscr{Q}_d(\eta_{\mathscr{A}}) = \mathrm{id}_{\mathscr{Q}_d(\mathscr{A})} = \mu^{1,d}_{\mathscr{A}} \circ \eta_{\mathscr{Q}_d(\mathscr{A})}$ and $\mu^{d,d'd''}_{\mathscr{A}} \circ \mathscr{Q}_d(\mu^{d',d''}_{\mathscr{A}}) = \mu^{dd',d''}_{\mathscr{A}} \circ \mu^{d,d'}_{\mathscr{Q}_{d''}(\mathscr{A})}$.

Proposition 3.1. *$((\mathscr{Q}_d)_{d\in\mathbb{N}}, \eta, (\mu^{d,d'})_{d,d'\in\mathbb{N}})$ is a graded monad defined on* **RStr**.

The full proof of this fact can be found in [1]. Also in [1], we have defined a quantum version of homomorphisms between relational structures, and shown that the Kleisli category of the quantum monad $\mathscr{Q}_d$ is (equivalent to) the category of relational structures with quantum homomorphism between them. Hence, by Theorem 16 from [1], in the case of graphs, we have that the existence of a Kleisli map $G \to \mathscr{Q}_d(H)$ implies the existence of a quantum graph homomorphism $G \to \mathsf{M}H$ in the sense of [31, 24, 25] (*i.e.* in the sense of Definition 3.3 above).

3.3 Quantum Maps of Relational Structures

We have observed that 'quantum' maps $\mathscr{A} \to \mathscr{Q}_d(\mathscr{B})$ between relational structures can be interpreted as projection-valued relations called quantum homomorphisms or, more operationally, as quantum perfect strategies for the non-local homomorphism game on relational structures from [1]. In this section we prove that the category of relational structures with quantum maps is an effectus. More exactly, we see that relational structures with maps $\mathscr{A} \to \mathscr{Q}_d(\mathscr{B})$ form an effectus.

Since the functor **RStr** $\to$ **Set** is a bifibration (see Remark 3.3) preserving finite products and coproducts, the category **RStr** of relational structures inherits the finite coproducts $(+,0)$ and (a choice of) a terminal object 1 from the category **Set** of sets. For instance, given $\mathscr{A},\mathscr{B} \in$ **RStr** relational structures, $\mathscr{A}+\mathscr{B} \in$ **RStr** is the relational structure over the set $A+B$ where the k-ary relation $R^{\mathscr{A}+\mathscr{B}}$ is defined as all the tuples $(x_1,\dots,x_k) \in (A+B)^k$ satisfying either $(x_1,\dots,x_k) \in R^{\mathscr{A}}$ or $(x_1,\dots,x_k) \in R^{\mathscr{B}}$. Also, we have the structure $0 \in$ **RStr** over the empty set $\emptyset = 0 \in$ **Set** with no relations. Further we have a structure $1 \in$ **RStr** over some singleton set $1 = \{*\}$ with the universal relation of arity k, *i.e.* one has $R^1 := 1^k = 1\times\cdots\times 1$.

Like the distribution monad $\mathscr{D}_M$, the quantum monad $\mathscr{Q}_d$ is also an affine monad since $\mathscr{Q}_d(1) \cong 1$. In fact, the functor $\mathscr{Q}_d$ is a lift of $\mathscr{D}_M$ along the fibration **RStr** $\to$ **Set** when $M = \mathrm{Proj}(d)$:

$$\begin{array}{ccc} \mathbf{RStr} & \xrightarrow{\mathscr{Q}_d} & \mathbf{RStr} \\ \downarrow & & \downarrow \\ \mathbf{Set} & \xrightarrow[\mathscr{D}_M]{} & \mathbf{Set} \end{array}$$

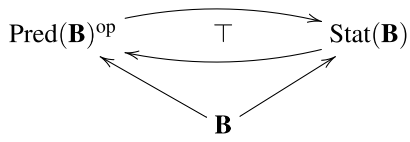
That is, $\mathscr{D}_M(A) = \mathscr{Q}_d(A)$ if $M = \mathrm{Proj}(d)$ for any $\mathscr{A} = (A, R^{\mathscr{A}}) \in$ **RStr** (see Definition 3.6). Hence $\mathscr{K}\ell(\mathscr{Q}_d)$ has a terminal object $1 \cong \mathscr{Q}_d(1) \in \mathscr{K}\ell(\mathscr{Q}_d)$ and finite coproducts $(+,0)$ given by disjoint union $+$ of relational structures and $0 \in \mathscr{K}\ell(\mathscr{Q}_d)$ the empty structure.

Theorem 3.1. *The Kleisli category $\mathscr{K}\ell(\mathscr{Q}_d)$ of the quantum monad $\mathscr{Q}_d$ is an effectus.*

Proof. We have already proved in Proposition 2.2 that the Kleisli category of the quantum monad on sets, which is a particular example of the distribution monad $\mathscr{D}_M$, is an effectus. That is, the two pullback conditions and one joint monicity requirement from Definition 2.4 hold for the Kleisli category $\mathscr{K}\ell(\mathscr{Q}_d)$ forgetting the homomorphism part (*i.e.* preserving relations). Thus, all that carries the same at the level of sets. Now we mention the parts about homomorphism. Let $u\colon \mathscr{P} \to \mathscr{Q}_d(\mathscr{A}+\mathscr{B})$ be the Kleisli map defined (in the first pullback condition) for each $a \in P$ as $u(a)(x) := c(a)(x) \in \mathrm{Proj}(d)$ for all $x \in A$, and $u(a)(y) := d(a)(y) \in \mathrm{Proj}(d)$ for all $y \in B$, where $c\colon \mathscr{P} \to \mathscr{Q}_d(\mathscr{A}+1)$ and $d\colon \mathscr{P} \to \mathscr{Q}_d(1+\mathscr{B})$ are given homomorphisms. This Kleisli map u is homomorphism by definition, since both c and d are homomorphisms by assumption. For the second pullback condition, now we suppose to have a homomorphism $c\colon \mathscr{P} \to \mathscr{Q}_d(\mathscr{A}+\mathscr{B})$ and define $u\colon \mathscr{P} \to \mathscr{Q}_d(\mathscr{A})$ as $u(a)(x) := c(a)(x)$ for all $a \in P$ and $x \in A$. Once again, here we have that u is homomorphism by definition since c is homomorphism by assumption. This completes the proof. □

3.4 States and Predicates, Validity and Conditioning

Essential aspects of the semantics of programs like state and predicate transformers are core parts of the internal logic of an effectus. Speaking intuitively, states are used for representing a state of affairs and predicates for representing evidence (classically, in the form of events). In the effectus obtained by taking the Kleisli category of the quantum monad $\mathscr{K}\ell(\mathscr{Q}_d)$, states are quantum measurments and predicates are 'quantum' events represented by assignments of orthogonal subspaces to possible outcomes. We shall start describing what is the situation with respect to states and predicates in general for an arbitrary effectus $\mathbf{B}$. A *state* on $X \in \mathbf{B}$ is a morphism in $\mathbf{B}$ with type $1 \to X$. A *predicate* on X is a morphism in $\mathbf{B}$ with type $X \to 1+1$. There is an adjunction:



where $\mathbf{B} \to \mathrm{Stat}(\mathbf{B})$ and $\mathbf{B} \to \mathrm{Pred}(\mathbf{B})^{\mathrm{op}}$ are the functors defined on objects as $\mathrm{Stat}(X) := \mathbf{B}(1,X)$ and $\mathrm{Pred}(X) := \mathbf{B}(X,1+1)$ for any $X \in \mathbf{B}$; the action of these functors on a given morphism $f\colon X \to Y$ in $\mathbf{B}$ produce morphisms $\mathrm{Stat}(f)\colon \mathrm{Stat}(X) \to \mathrm{Stat}(Y)$ and $\mathrm{Pred}(f)\colon \mathrm{Pred}(Y) \to \mathrm{Pred}(X)$ called *state* and *predicate transformer* defined by compositon in $\mathbf{B}$ as $\mathrm{Stat}(f)(\omega) := f \circ \omega$ and $\mathrm{Pred}(f)(q) := q \circ f$, for any $\omega \in \mathrm{Stat}(X)$ and $q \in \mathrm{Pred}(Y)$. The following notation is standard in the field:

$$f \gg \omega := \mathrm{Stat}(f)(\omega) \in \mathrm{Stat}(Y) \qquad\qquad f \ll q := \mathrm{Pred}(f)(q) \in \mathrm{Pred}(X)$$

With this terminology and notation, we have that state transformer acts forwardly while predicate transformer acts backwardly. Given a state $\omega \in \mathrm{Stat}(X)$ and a predicate $p \in \mathrm{Pred}(X)$ on the same object $X \in \mathbf{B}$, the *validity* $\omega \models p$ of p in ω is defined to be the morphism $p \circ \omega \in \mathbf{B}(1,1+1)$. Morphisms in $\mathbf{B}(1,1+1)$ are called *scalars*. Hence, validity is a scalar representing the degree of certainty of some evidence in the current state of affairs.

The concept of 'channel' is central in the logical essentials of effect-theoretic reasoning and its applications, see *e.g.* [4, 21, 18]. Formally, a *channel* is just a morphism in some effectus. For the category **Set**, channels are functions. For discrete probabilities $\mathscr{K}\ell(\mathscr{D}_M)$, channels are conditional probability distributions $X \to \mathscr{D}_M(Y)$. Finally, for quantum probabilities $\mathscr{K}\ell(\mathscr{Q}_d)$, channels are quantum homomorphisms of relational structures $\mathscr{A} \to \mathscr{Q}_d(\mathscr{B})$. The effectus $\mathscr{K}\ell(\mathscr{Q}_d)$ has relational structures as objects and homomorphisms of type $\mathscr{A} \to \mathscr{Q}_d(\mathscr{B})$ as morphisms. At the level of sets, states of $\mathscr{K}\ell(\mathscr{Q}_d)$ are quantum measurements (*i.e.* projection valued distributions) and since $\mathscr{Q}_d(2) \cong \mathrm{Proj}(d)$, predicates are assignments of projectors. We can think about $\mathrm{Proj}(d)$ as a relational structure of projectors with a k-ary relation $R^{\mathrm{Proj}(d)}$ given by $(p_1,\ldots,p_k) \in R^{\mathrm{Proj}(d)}$ if and only if $p_i \cdot p_j = p_j \cdot p_i$ for all $i,j = 1,\ldots,k$.

Proposition 3.2. *Let $\mathscr{A} = (A, R^{\mathscr{A}}) \in \mathbf{RStr}$ be a relational structure. Then:*

- *a* state *on $\mathscr{A}$ is a quantum (projective) measurement $p \in \mathscr{Q}_d(\mathscr{A})$ on the underlying set A;*
- *a* predicate $f\colon \mathscr{A} \to \mathrm{Proj}(d)$ *is an assignment of projectors $f\colon A \to \mathrm{Proj}(d)$ such that points appearing in some tuple in the relation $R^{\mathscr{A}}$ get assigned commuting projectors, i.e. projectors $f(x_i), f(x_j) \in \mathrm{Proj}(d)$ commute if there exists $\alpha \in A^k$ such that $\alpha = (x_1,\ldots,x_i,\ldots,x_j,\ldots,x_k)$ and $\alpha \in R^{\mathscr{A}}$.*

Given a state and a predicate we can always compute their validity, or expected value, of the predicate in the state.

Proposition 3.3. *Let $p \in \mathscr{Q}_d(\mathscr{A})$ be a state and $f\colon \mathscr{A} \to \mathrm{Proj}(d')$ be a predicate. The* validity $p \models f \in \mathrm{Proj}(dd')$ *of the predicate f in the measurement p is the projector given by Kleisli composition $f_* \circ p$:*

$$p \models f := \sum_{x \in A} p(x) \otimes f(x)$$

Now describe conditional or evidential reasoning.

Proposition 3.4. *Given a state $p \in \mathscr{Q}_d(\mathscr{A})$ and a predicate $f\colon \mathscr{A} \to \mathrm{Proj}(d')$ on the same structure $\mathscr{A}$,* conditioning p given f *is the state $p|_f \in \mathscr{Q}_{dd'}(\mathscr{A})$ defined if validity $p \models f$ is non-zero as the formal convex sum:*

$$p|_f := \sum_{x \in A} \frac{p(x) \otimes f(x)}{p \models f} |x\rangle$$

4 Outlook

In this paper we expounded some details regarding quantum and probabilistic reasoning via the notion of *effectus* from categorical logic [15]. The exposition has build on previous work [1] about the logical and categorical structure of quantum solutions to constraint systems via the quantum monad $\mathscr{Q}_d$. Here it has been shown that the Kleisli category $\mathscr{K}\ell(\mathscr{Q}_d)$ of the quantum monad forms an effectus. States are quantum measurements (*i.e.* projection-valued distributions) and predicates are assignments of d-dimensional projections to elements. In case of having relations and not just sets, predicates must assign commuting projections to elements in some tuple in some relation. Other aspects of the semantics of programs like channels, validity and conditioning, which are core parts of the internal logic of an effectus were also instantiated in the effectus $\mathscr{K}\ell(\mathscr{Q}_d)$.

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