

Circuit Relations for Real Stabilizers: Towards TOF + H

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The real stabilizer fragment of quantum mechanics was shown to have a complete axiomatization in terms of the angle-free fragment of the ZX-calculus. This fragment of the ZX-calculus—although abstractly elegant—is stated in terms of identities, such as spider fusion which generally do not have interpretations as circuit transformations.

We complete the category CNOT generated by the controlled not gate and the computational ancillary bits, presented by circuit relations, to the real stabilizer fragment of quantum mechanics. This is performed first, by adding the Hadamard gate and the scalar $\sqrt{2}$ as generators. We then construct translations to and from the angle-free fragment of the ZX-calculus, showing that they are inverses. We remove the generator $\sqrt{2}$ and then prove that the axioms are still complete for the remaining generators. This yields a category which is not compact closed, where the yanking identities hold up to a non-invertible, non-zero scalar.

We then discuss how this could potentially lead to a complete axiomatization, in terms of circuit relations, for the approximately universal fragment of quantum mechanics generated by the Toffoli gate, Hadamard gate and computational ancillary bits.

1 Background

The angle-free fragment of the ZX calculus—describing the interaction of the Z and X observables, Hadamard gate and π phases—is known to be complete for (pure) real stabilizer circuits (stabilizer circuits with real coefficients) [13]. In [13, Section 4.1], it is shown that real stabilizer circuits are generated by the controlled- Z gate, the Z gate, the Hadamard gate, $|0\rangle$ state preparations and $\langle 0|$ post-selected measurements. Therefore, real stabilizer circuits can also be generated by the controlled-not gate, Hadamard gate, $|1\rangle$ state preparations and $\langle 1|$ post-selected measurements. Although the Hadamard gate, controlled-not gate and computational ancillary bits are derivable in the angle-free fragment of the ZX-calculus, the identities are not given in terms of circuit relations involving these gates: and instead, on identities such as spider fusion which do not preserve the causal structure of being a circuit. Therefore circuit simplification usually involves a circuit reconstruction step at the end. Moreover, as a consequence of spider-fusion and in particular, the (co)unitality of the spiders, certain states in the ZX-calculus are not stabilizer states; as they have norms greater than 1.

We provide a complete set of *circuit identities* for the category generated by the controlled-not gate, Hadamard gate and state preparation for $|1\rangle$ and postselected measurement for $\langle 1|$. The completeness is proven by performing a translation to and from the angle-free fragment of the ZX-calculus; however, in contrast to the ZX-calculus, we structure the identities so that they preserve the causal structure of circuits. Although the axiomatization we describe is just as computationally expressive as the angle-free fragment of the ZX-calculus, it provides a high-level language for real stabilizer circuits; and hopefully, alongside the circuit axiomatization for Toffoli circuits, will lead to a high-level language for Hadamard+Toffoli circuits.

The compilation of quantum programming languages can involve multiple intermediate steps before an optimized physical-level circuit is produced; where optimization can be performed at various levels of granularity [20]. At the coarsest level of granularity, classical oracles and subroutines are synthesized using generalized controlled-not gates. Next these generalized controlled-not gates are decomposed into cascading Toffoli gates, and so on... Eventually, these gates are decomposed into fault tolerant 1 and 2-qubit gates.

The ZX-calculus has proven to be successful for this fine grain optimization, in particular at reducing T counts [19]. This optimization is performed in three steps: first, a translation must be performed turning the circuit into spiders and phases. Second, the spider fusion laws, Hopf laws, bialgebra laws and so on are applied to reduce the number of nodes/phases; transforming the circuit into a simpler form resembling an undirected, labeled graph without a global causal structure. Finally, an optimized circuit is re-extracted from this undirected graph. In order to extract circuits at the end, for example, [11, 12] use a property of graphs called gFlow.

Using only circuit relations, in contrast, [14] were able to reduce 2-qubit Clifford circuits to minimal forms in quantomatic.

Toffoli+Hadamard quantum circuits, as opposed to the ZX-calculus, are more suitably a language for classical oracles, and thus, are appropriate for coarse granularity optimization. The controlled-not+Hadamard subfragment, on the other hand, which we discuss in this paper one can only produce oracles for affine Boolean functions—which is obviously very computationally weak. The eventual goal, however, is to use this complete axiomatization controlled-not+Hadamard circuits given in this paper, and the axiomatization of Toffoli circuits provided in [8], to provide a complete set of identities for the approximately universal [2] fragment Toffoli+Hadamard circuits. In this fragment, indeed, all oracles for classical Boolean functions can be constructed [8, 1]. In Section 5, we discuss how this circuit axiomatization of controlled-not+Hadamard circuits could potentially lead to one for Toffoli+Hadamard circuits,

Toffoli+Hadamard circuits also easily accommodate the notion of quantum control. This is useful for implementing circuits corresponding to the conditional execution of various subroutines; which is discussed in [15, Section 2.4.3] and [17]. Although, in the fragment which we discuss in this paper, we can not control all unitaries: namely circuits containing controlled-not gates can not be controlled. Again, the eventual goal is to extend the axiomatizations of cnot+Hadamard and Toffoli circuits to Toffoli+Hadamard circuits, where there is no such limitation.

In the ZX-calculus, by contrast, this notion of control is highly unnatural. One would likely have to appeal to the triangle gate, as discussed in [21, 18, 22].

2 The controlled not gate

Recall that CNOT is the PROP generated by the 1 ancillary bits $|1\rangle$ and $\langle 1|$ as well as the controlled not gate:

$$|1\rangle := \text{---} \quad |1\rangle := \text{---} \quad \text{cnot} := \text{---}$$

Where “gaps” are drawn between cnot gates and cnot gates are drawn upside down to suppress symmetry maps:

$$\text{---} := \text{---} \quad \text{---} := \text{---}$$

These gates must satisfy the identities given in Figure 1:

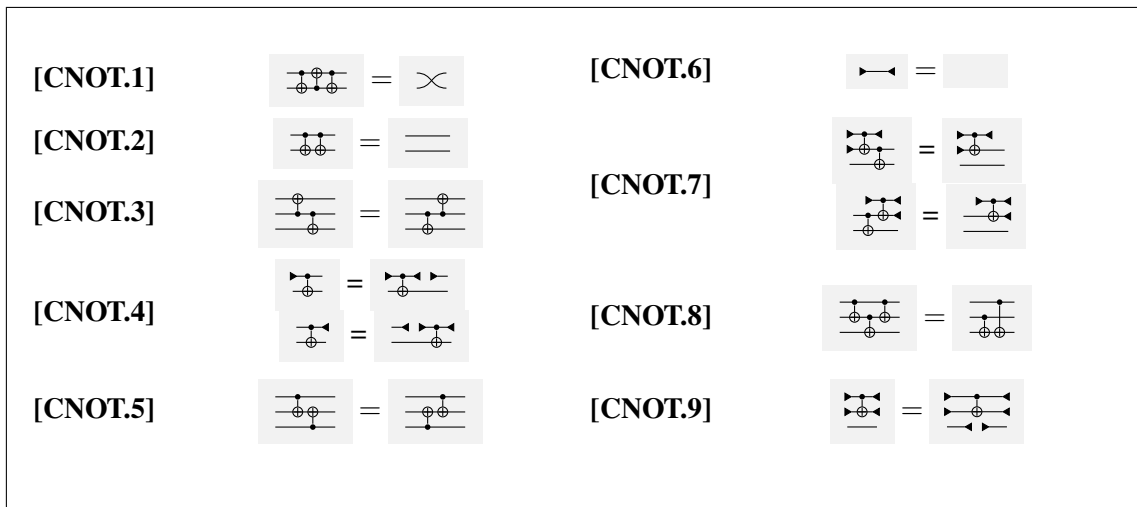


Figure 1: The identities of CNOT

Where the not gate and $|0\rangle$ ancillary bits are derived:

$$\text{not} = \text{---} := \text{---} \quad |0\rangle = \text{---} := \text{---} \quad \langle 0| = \text{---} := \text{---}$$

Where there is a pseduo-Frobenius structure (non-unital classical structure) generated by:

$$\text{---} := \text{---} \quad \text{and} \quad \text{---} := \text{---}$$

There is the following completeness result:

Theorem 2.1. CNOT is discrete-inverse equivalent to the category of affine partial isomorphisms between finite-dimensional \mathbb{Z}_2 vector spaces, and thus, is complete.

3 Stabilizer quantum mechanics and the angle-free ZX-calculus

In this section, we briefly describe the well known fragment of quantum mechanics known as stabilizer quantum mechanics. In particular we focus on the real fragment of stabilizer mechanics, and describe a complete axiomization thereof called the angle-free ZX-calculus. Stabilizer quantum mechanics are very well studied, a good reference from a categorical perspective is given in [4].

Definition 3.1. The **Pauli matrices** are the complex matrices:

$$X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The **Pauli group** on n is the closure of the set:

$$P_n := \{\lambda a_1 \otimes \cdots \otimes a_n \mid \lambda \in \{\pm 1, \pm i\}, a_i \in \{I_2, X, Y, Z\}\}$$

under matrix multiplication.

The **stabilizer group** of $|\varphi\rangle$ denoted by $S_{|\varphi\rangle}$ a quantum state is the group of operators for which $|\varphi\rangle$ is a +1 eigenvector. A state is a **stabilizer state** in case it is stabilized by a subgroup of P_n .

Definition 3.2. The **Clifford group** on n is the group of operators which acts on the Pauli group on n by conjugation:

$$C_n := \{U \in U(2^n) \mid \forall p \in P_n, U p U^{-1} \in P_n\}$$

There is an algebraic description of stabilizer states:

Lemma 3.3. All n qubit stabilizer states have the form $C|0\rangle^{\otimes n}$, for some member C of the Clifford group on n qubits.

Indeed, we also consider a subgroup of C_n :

Definition 3.4. The **real Clifford group** on n qubits, is the subgroup of the Clifford group with real elements, ie:

$$C_n^{re} := \{U \in C_n \mid \overline{U} = U\}$$

So that an n -qubit **real stabilizer state** is a state of the form $C|0\rangle^{\otimes n}$ for some real Clifford operator C .

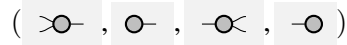
We say that a **(real) stabilizer circuit** is a (real) Clifford composed with state preparations and measurements in the computational basis.

The **ZX-calculus** is a collection of calculi describing the interaction of the complementary Frobenius algebras corresponding to the Pauli Z and X observables and their phases. The stabilizer fragment of the ZX-calculus restricts the angles of phases to multiples of $\pi/2$. However, we are interested in an even simpler fragment of the ZX-calculus, namely the angle-free calculus for real stabilizer circuits described in [13] (slightly modified to account for scalars):

Definition 3.5. Let ZX_π denote the \dagger -compact closed PROP with generators:



such that



is a classical structure and the following identities also hold up to swapping colours:

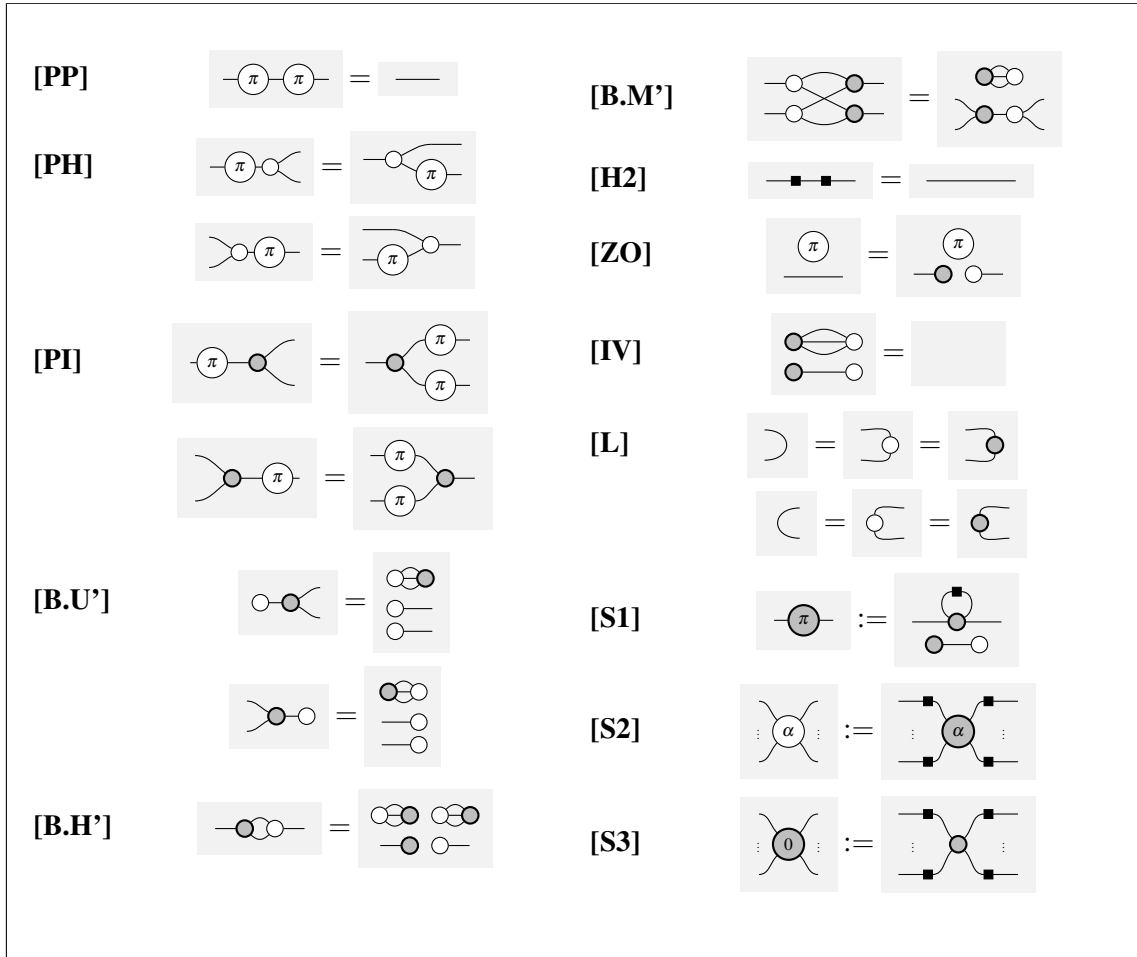
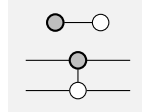


Figure 2: The identities of ZX_π (where $\alpha \in \{0, \pi\}$)

Where the last 3 Axioms are actually definitions, which simplify the presentation of ZX_π . Note that the axioms of a classical structure are omitted from this box to save space.

This category has a canonical \dagger -functor, as all of the stated axioms have are horizontally symmetric. In fact, it is \dagger -compact closed. This category embeds faithfully in FHilb; the black Frobenius algebra corresponds to the Pauli Z basis; the white Frobenius algebra corresponds to the Pauli X basis; the gate $\text{---}\blacksquare\text{---}$ corresponds to the Hadamard gate and $\text{---}\pi\text{---}$ and $\text{---}\pi\text{---}$ correspond to Z and X π -phase-shifts respectively. In particular, the X π -phase-shift is the not gate.

Note that the controlled-not gate has a succinct representation in ZX_π (this can be verified by calculation):



This means that ZX_π contains all of the generators of the real Clifford group; furthermore, the identities are complete in the sense that:

Theorem 3.6. [13] ZX_π is complete for real stabilizer states.

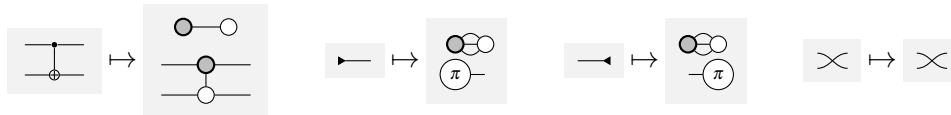
The original presentation of ZX_π in [13] did not account for scalars; instead, it imposed the equivalence relation on circuits up to an invertible scalar and ignored the zero scalar entirely. Therefore, the completeness result described in [13] is not actually as strong as the aforementioned theorem, as stated. This means, of course, that this original calculus does not embed in FHilb as the relations are not sound. For example, the following map is interpreted as $\sqrt{2}$, not 1 in FHilb:



Later on, [3, 6] showed that by scaling certain axioms to make them sound, and by adding Axioms **[IV]** and **[ZO]** this fragment of the ZX -calculus is also complete for scalars. We, of course, have done the proper scaling, and added the aforementioned Axioms so that such a full completeness result can be enjoyed.

4 The controlled-not gate plus the Hadamard gate

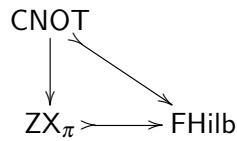
Consider the interpretation of CNOT into ZX_π , sending:



We have:

Lemma 4.1. The interpretation of CNOT into ZX_π is functorial.

The following diagram commutes, making this functor $\text{CNOT} \rightarrow \text{ZX}_\pi$ faithful:



To extend CNOT to the real fragment of Clifford quantum mechanics, we have to add one gate, a scalar, and some identities:

Definition 4.2. Let $\text{CNOT} + H$ denote the PROP freely generated by the axioms of CNOT with additional generators the Hadamard gate and $\sqrt{2}$:



satisfying:

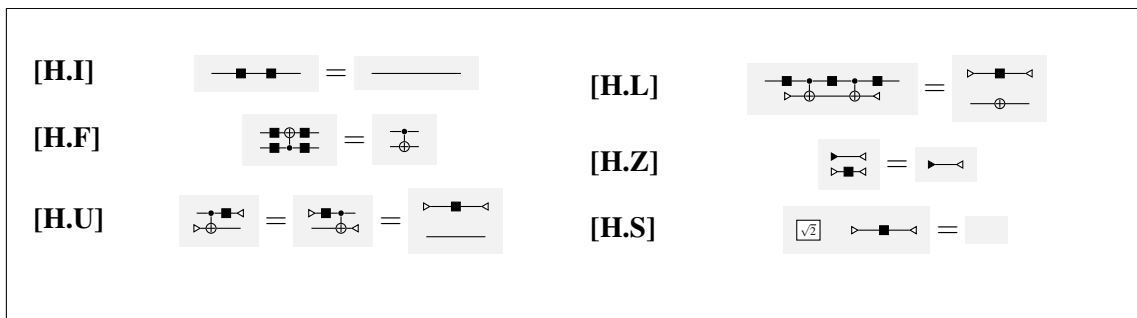
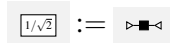
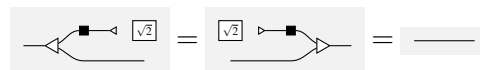


Figure 3: Identities of $\text{CNOT} + H$ (in addition to the identities of CNOT)

Where we give the inverse of $\sqrt{2}$ the alias:



[H.I] is stating that the Hadamard gate is self-inverse. **[H.F]** reflects the fact that composing the controlled-not gate with Hadamards reverses the control and operating bits. **[H.Z]** is stating that $1/\sqrt{2}$ composed with the zero matrix is again the zero matrix. **[H.S]** makes $\sqrt{2}$ and $1/\sqrt{2}$ inverses to each other. **[H.U]**, can be restated, using Δ , Δ° and **[H.S]**, as completing the unit of Δ and the counit of Δ° :



[H.L], is a translation of [S1] (the H-loop law), and can be restated, using Δ , Δ° , [H.S] and [H.I] as follows:

We construct two functors between $\text{CNOT} + H$ and ZX_π and show that they are pairwise inverses.

Definition 4.3. Let $F : \text{CNOT} + H \rightarrow \text{ZX}_\pi$, be the extension of the interpretation $\text{CNOT} \rightarrow \text{ZX}_\pi$ which takes:

Let $G : \text{ZX}_\pi \rightarrow \text{CNOT} + H$ be the interpretation sending:

These interpretations are functors:

Lemma 4.4. $F : \text{CNOT} + H \rightarrow \text{ZX}_\pi$ is a functor.

For the other way around, we have

Lemma 4.5. $G : \text{ZX}_\pi \rightarrow \text{CNOT} + H$ is a functor.

Proposition 4.6. $\text{CNOT} + H \xrightarrow{F} \text{ZX}_\pi$ and $\text{ZX}_\pi \xrightarrow{G} \text{CNOT} + H$ are inverses.

Because all of the axioms of $\text{CNOT} + H$ and ZX_π satisfy the same ‘‘horizontal symmetry’’; we can not only conclude that they are isomorphic, but rather:

Theorem 4.7. $\text{CNOT} + H$ and ZX_π are \dagger -isomorphic.

Because scalars correspond to probabilities; it is natural to ask if we can remove the scalar $\sqrt{2}$ from $\text{CNOT} + H$ to obtain a calculus where all circuits correspond to actual quantum protocols.

Definition 4.8. Let $\overline{\text{CNOT} + H}$ denote the subcategory of $\text{CNOT} + H$ without the generator $\sqrt{2}$ (and thus without Axiom [H.S]).

Corollary 4.9. The identities of $\overline{\text{CNOT} + H}$ are complete.

Proof. Consider the category $\text{CNOT} + H$ quotiented out by nonzero scalars, where [H.S] is rendered trivial. Remark that this category is complete in the sense that two circuits which are equal up to a nonzero scalar can be reduced to one and other. We then add the scalars back in, without $\sqrt{2}$ or [H.S], to obtain $\overline{\text{CNOT} + H}$. Remark that in this category, two circuits which are equal up to a nonzero scalar can be shown to be reducible to each other up to a nonzero scalar; the question is, can this scalar be shown to be unique?

Therefore, to show that $\overline{\text{CNOT} + H}$ is complete, we must show the completeness for nonzero scalars. This kind of argument is very similar to that which was used to show the completeness of the ZX-calculus with scalars in [3].

Remark that by [3, Proposition 5] all nonzero real stabilizer scalars are of the form $\pm 2^{n/2}$ for some integer n . Therefore, any non-zero scalar in $\text{CNOT} + H$ is in the closure of the group:

$$\langle \sqrt{2}, 1/\sqrt{2}, -1 \rangle$$

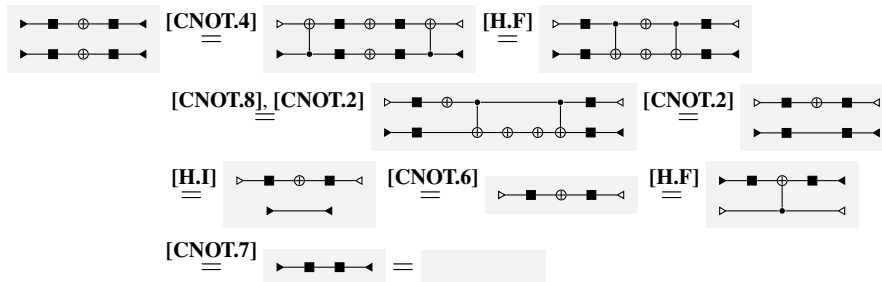
Where -1 denotes the following circuit, which can be verified to have the interpretation -1 in FHilb:



Therefore, nonzero scalars in $\overline{\text{CNOT} + H}$ are in the closure of the monoid:

$$\langle -1, 1/\sqrt{2} \rangle$$

We already know that 1 is the identity; therefore, it suffices to show that $-1 \cdot -1 = 1$ in $\overline{\text{CNOT} + H}$:



So that $1/\sqrt{2} \cdot 1/\sqrt{2} = -1/\sqrt{2} \cdot -1/\sqrt{2}$. Therefore, nonzero scalars have a normal form and are unique. □

Remark that $\overline{\text{CNOT} + H}$, like ZX_π , is complete for real stabilizer quantum mechanics; however, unlike ZX_π , every scalar in $\overline{\text{CNOT} + H}$ lies within the (real) unit ball; and thus, when taking the square, can be interpreted as a probability. Indeed, the circuits in $\overline{\text{CNOT} + H}$ do not increase entropy, and thus, correspond to (possibly noisy) quantum protocols. This difference is attributed to the fact that in the ZX calculus, the spider fusion law forces the units of the Frobenius algebras to be unnormalized.

5 Towards the Toffoli gate plus the Hadamard gate

Recall the PROP TOF, generated by the 1 ancillary bits $|1\rangle$ and $\langle 1|$ (depicted graphically as in CNOT) as well as the Toffoli gate:

$$\text{tof} := \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ \oplus \end{array}$$

The axioms are given Figure 4, which we have put in Appendix G. Recall that have that:

Theorem 5.1. TOF is discrete-inverse equivalent to the category of partial isomorphisms between finite powers of the two element set, and thus, is complete.

By [1], we have that the Toffoli gate is universal for classical reversible computing, therefore TOF is a complete set of identities for the universal fragment of classical computing. However, the category is clearly is not universal for quantum computing. Surprisingly, by adding the Hadamard gate as a generator, this yields a category which is universal for an approximately universal fragment of quantum computing [2].

Thus, one would hope that the completeness of $\text{CNOT} + H$ could be used to give a complete set of identities for a category $\text{TOF} + H$.

Although we have not found such a complete set of identities, the identity **[H.F]** can be easily extended to an identity that characterizes the commutativity of a multiply controlled-Z-gate. This could possibly facilitate a two way translation to and from the ZH calculus [5], like we performed between $\text{CNOT} + H$ and ZX_π . This, foreseeably would be much easier than a translation between one of the universal fragments of the ZX-calculus; because, despite the recent simplifications of the Toffoli gate in terms of the triangle, the triangle itself does not have a simple representation in terms of the Toffoli gate, Hadamard gate and computational ancillary bits [22].

If we conjugate the not gate (X gate) with Hadamard gates, we get the Z gate:

$$\text{---} \bullet \text{---} := \text{---} \blacksquare \oplus \blacksquare \text{---}$$

Furthermore, if we conjugate the operating bit controlled-not gate with the the Hadamard gate, we get the controlled-Z gate:

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ \oplus \end{array} := \text{---} \blacksquare \oplus \blacksquare \text{---}$$

Because the flow of information in the controlled-Z gate is undirected in the sense that:

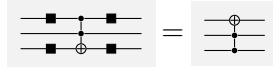
$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ \oplus \end{array} = \begin{array}{c} \text{---} \\ \times \\ \text{---} \\ \times \\ \text{---} \end{array}$$

this motivates the identity **[H.F]** of $\text{CNOT} + H$:

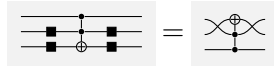
$$\begin{array}{c} \text{---} \blacksquare \oplus \blacksquare \text{---} \\ | \\ \text{---} \blacksquare \oplus \blacksquare \text{---} \\ | \\ \text{---} \blacksquare \oplus \blacksquare \text{---} \end{array} = \begin{array}{c} \text{---} \blacksquare \oplus \blacksquare \text{---} \\ | \\ \text{---} \blacksquare \oplus \blacksquare \text{---} \\ | \\ \text{---} \blacksquare \oplus \blacksquare \text{---} \end{array} = \begin{array}{c} \text{---} \blacksquare \oplus \blacksquare \text{---} \\ \times \\ \text{---} \blacksquare \oplus \blacksquare \text{---} \\ \times \\ \text{---} \blacksquare \oplus \blacksquare \text{---} \end{array} = \begin{array}{c} \text{---} \blacksquare \oplus \blacksquare \text{---} \\ \times \\ \text{---} \blacksquare \oplus \blacksquare \text{---} \\ \times \\ \text{---} \blacksquare \oplus \blacksquare \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ \oplus \end{array}$$

We can continue this, so that by conjugating the operating bit of the Toffoli gate with Hadamard gates, we obtain a doubly controlled-Z gate. This suggests the following (sound) identity:

[H.F']



Along with [TOF.16], this entails:



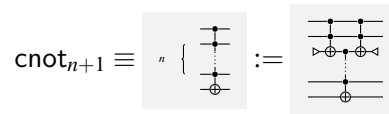
So that we can unambiguously represent the doubly controlled-Z gate as:



Indeed, this identity entails a more general form for generalized controlled-not gate with two or more controls. Recall the definition of a multiply controlled-not gate in [8]:

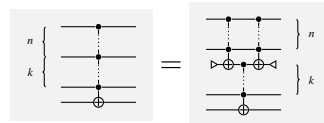
Definition 5.2. [8, Definition 5.1] For every $n \in \mathbb{N}$, inductively define the controlled not gate, $\text{cnot}_n : n + 1 \rightarrow n + 1$ inductively by:

- For the base cases, let $\text{cnot}_0 := \text{not}$, $\text{cnot}_1 := \text{cnot}$ and $\text{cnot}_2 := \text{tof}$.
- For all $n \in \mathbb{N}$ such that $n \geq 2$:



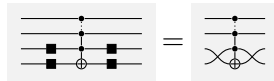
Recall cnot_n gates can be decomposed into other cnot_n gates in the following fashion:

Proposition 5.3. [8, Proposition 5.3 (i)] cnot_{n+k} gates can be zipped and unzipped:

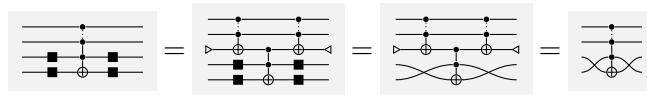


Therefore, can can derive that:

Lemma 5.4. [H.F'] entails:

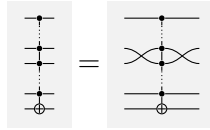


Proof. From the zipper lemma and [H.F'], we have:



□

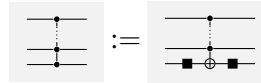
Recall from [8, Corollary 5.4] that control-wires of cnot_n gates can be permuted in the following sense:



Therefore, by this observation and Lemma 5.4 we have:



So that, by observing that the multiply controlled-Z gate can be unambiguously defined as follows:



Acknowledgement

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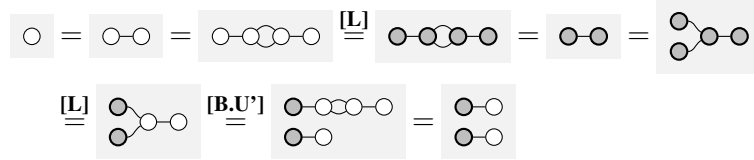
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A Identities

Lemma A.1. [22, Lemma 19] $\circlearrowleft = \begin{array}{c} \bullet\circ \\ \bullet\circ \end{array}$

Proof.

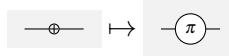


□

We also need:

Lemma A.2.

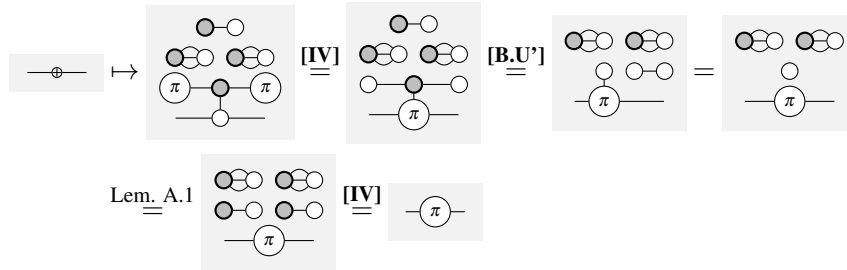
(i)



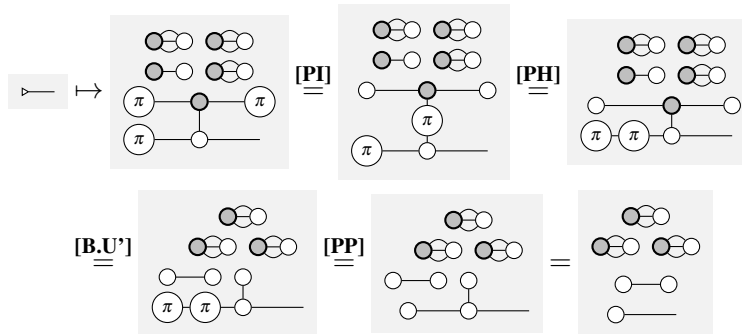
(ii)



Proof. (i)



(ii)



$$\text{Lem. A.1, [IV]} \quad \begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} = \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array}$$

□

B Proof of Lemma 4.1

Recall the statement of Lemma 4.1:

The interpretation of CNOT into ZX_π is functorial.

Proof. We prove that each axiom holds

[CNOT.2]

$$\begin{array}{c} \text{---} \oplus \text{---} \\ \text{---} \oplus \text{---} \end{array} \mapsto \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} = \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} \stackrel{[\text{B.H}']}{=} \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} \\ = \begin{array}{c} \text{---} \\ \text{---} \end{array} \leftarrow \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

[CNOT.1]

$$\begin{array}{c} \text{---} \oplus \text{---} \\ \text{---} \oplus \text{---} \end{array} \mapsto \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} = \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} \stackrel{[\text{B.M}']}{=} \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} \\ = \begin{array}{c} \text{---} \\ \text{---} \end{array} \leftarrow \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

[CNOT.3] Immediate from commutative spider theorem.

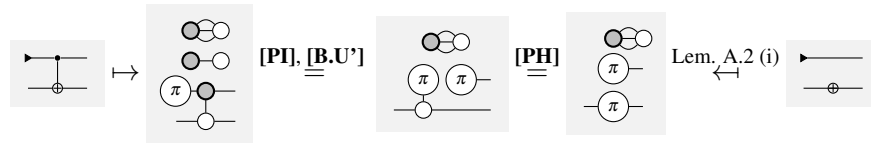
[CNOT.4]

$$\begin{array}{c} \text{---} \oplus \text{---} \\ \text{---} \oplus \text{---} \end{array} \xrightarrow{\text{Lem. A.2 (ii)}} \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} \stackrel{[\text{B.U}']}{=} \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} = \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} \xleftarrow{\text{Lem. A.2 (ii)}} \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

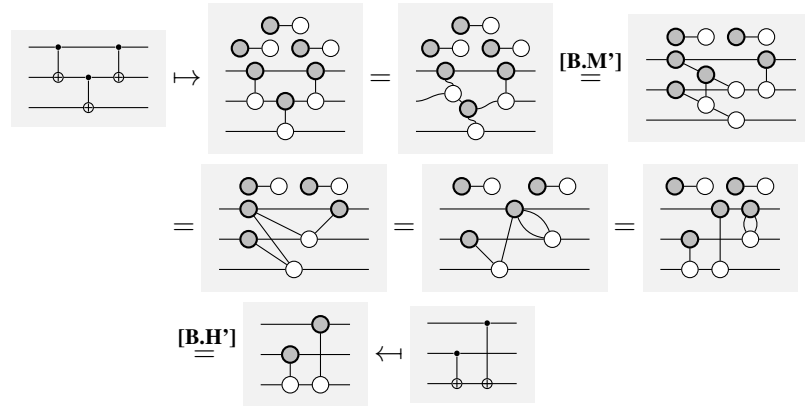
[CNOT.5] Immediate from commutative spider theorem.

[CNOT.6] Frobenius algebra is special.

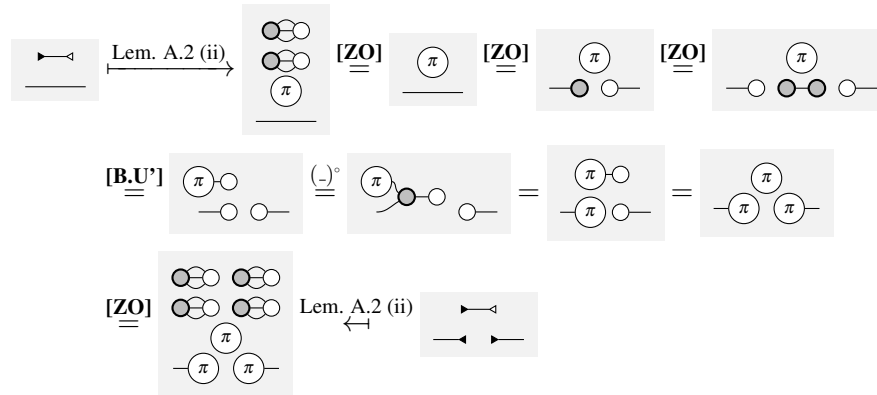
[CNOT.7]



[CNOT.8]



[CNOT.9]

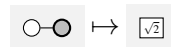


□

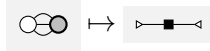
C More lemmas

Lemma C.1.

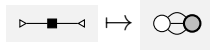
(i)



(ii)

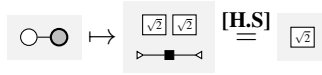


(iii)

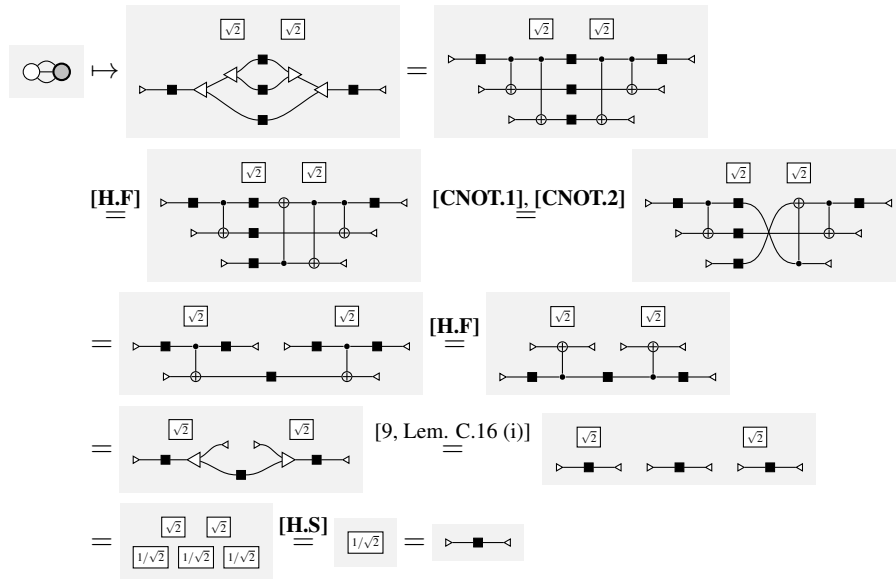


Proof.

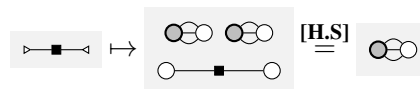
(i)



(ii)



(iii)



□

D Proof of Lemma 4.4

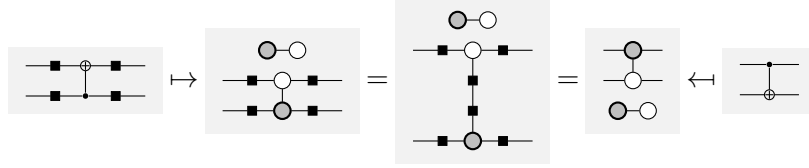
Recall the statement of Lemma 4.4:

$F : \text{CNOT} + H \rightarrow ZX_\pi$ is a functor.

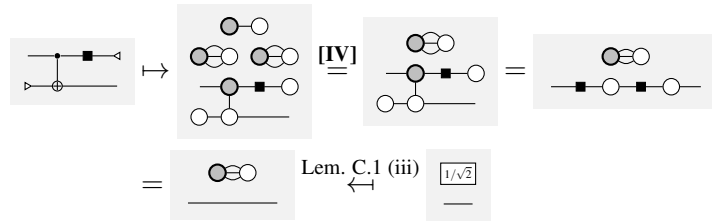
Proof. As the restriction of F to CNOT is a functor, it suffices to show that [H.I], [H.F], [H.U], [H.L], [H.S] and [H.Z] hold.

[H.I] Immediate from [H2].

[H.F]



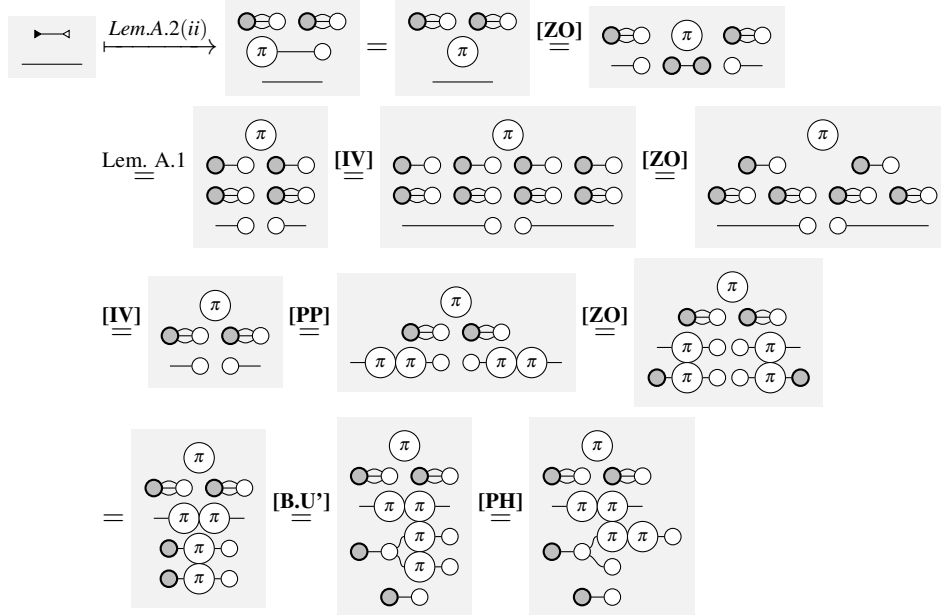
[H.U]

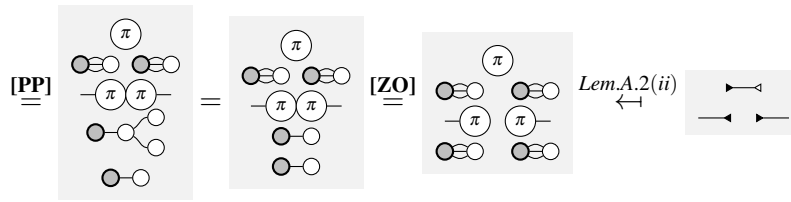


[H.L] Immediate from [S1].

[H.S] Immediate from [IV].

[H.Z]





□

E Proof of Lemma 4.5

Recall the statement of Lemma 4.5:

$G : ZX_\pi \rightarrow \text{CNOT} + H$ is a functor.

Proof. We prove that each axiom holds:

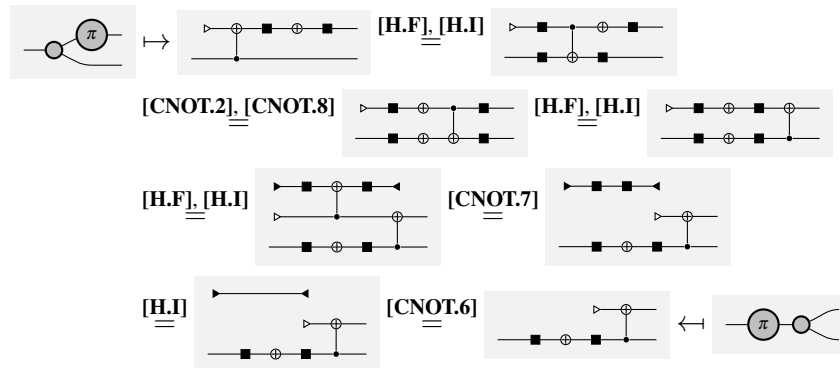
[PI] This follows by naturality of Δ .

[B.U'] This follows by naturality of Δ and [H.S].

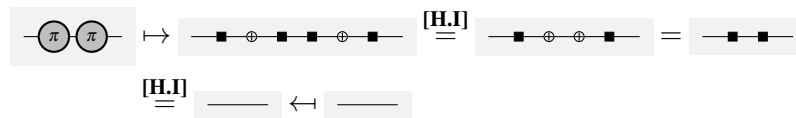
[H2] This follows immediately from [H.I].

[H2] This follows immediately from [H.S].

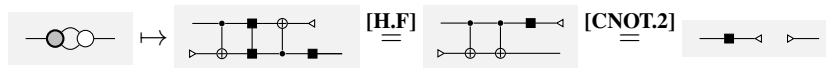
[PH]



[PP]



[B.H']



Classical structure: Remark that rules [H.U] and [H.S] complete the semi-Frobenius structure to the appropriate classical structure.

□

F Proof of Proposition 4.6

Recall the statement of Proposition 4.6:

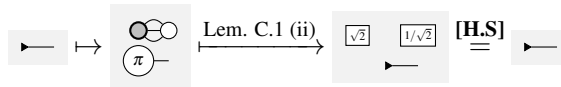
$$\text{CNOT} + H \xrightarrow{F} ZX_\pi \text{ and } ZX_\pi \xrightarrow{G} \text{CNOT} + H \text{ are inverses.}$$

Proof.

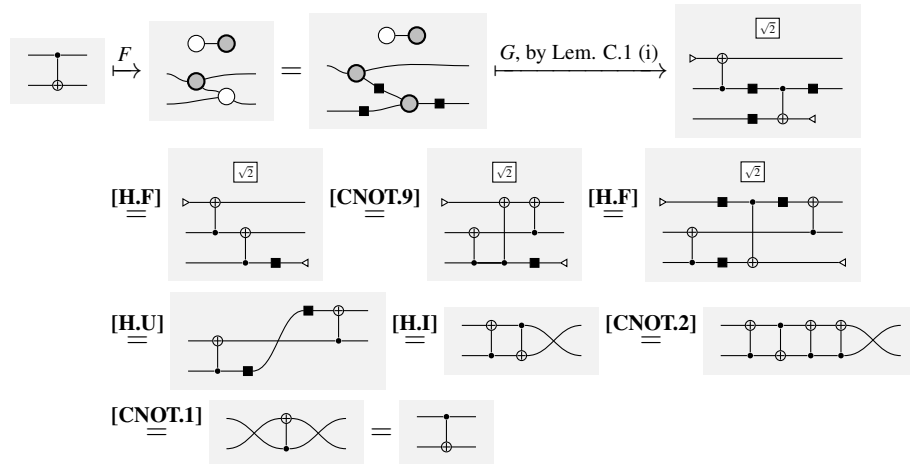
1. First, we show that $GF = 1$.

We only prove the cases for the generators cnot and $|1\rangle$ as the claim follows trivially for the Hadamard gate and by symmetry for $\langle 1|$:

For $|1\rangle$:



For cnot:

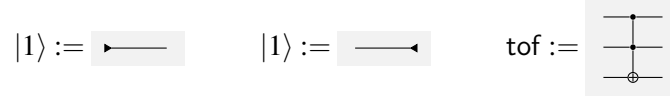


2. It is trivial to observe that $GF = 1$.

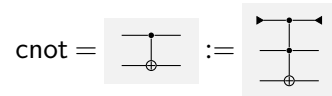
□

G The identities of TOF

Recall that TOF is the PROP generated by the 1 ancillary bits $|1\rangle$ and $\langle 1|$ as well as the Toffoli gate:



Where there is the derived generator



and the not gate and $|0\rangle$ ancillary bits are derived as in Section 2. These generators must satisfy the identities given in the following figure:

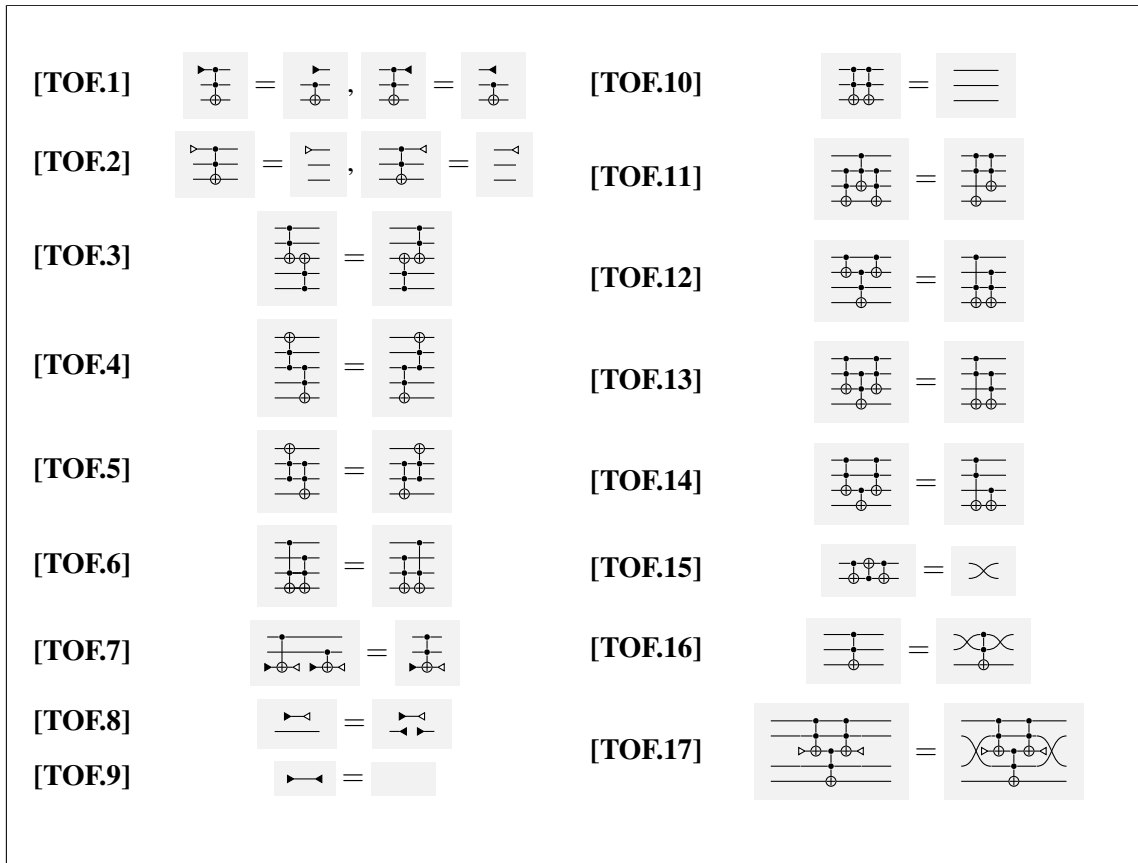


Figure 4: The identities of TOF