Structured cospans and the ZX-calculus

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In this talk, we are combining three approaches into a single formalism. Coecke and Duncan [3] introduced the ZX-calculus, a compositional graphical syntax, to reason about computations in multi-qubit systems. Baez and Courser [1] introduced structured cospans as a syntactical device to reason about compositional systems. The author introduced rewriting structured cospans [2]. Here, we illustrate rewriting structured cospans with the ZX-calculus. As a result, we obtain a symmetric monoidal double category $\mathbb{ZX}$ where the ZX-diagrams form the horizontal 1-arrows and the rewrites of ZX-diagrams form the 2-arrows. Moreover, we show that the horizontal bicategory of $\mathbb{ZX}$ is a bicategory of relations.

Recall the ZX-calculus is a compositional diagrammatic language generated by the ZX-diagrams depicted in Figure 1. New ZX-diagrams are created by the generating ZX-diagrams by connecting along dangling strings and also by the disjoint union of diagrams. The ZX-diagrams are then subjected to the equations listed in Figure 2

To define structured cospans, we begin with a geometric morphism

$$\mathsf{X} \underset{R}{\overset{L}{\leftrightarrows}} \mathsf{A} \quad (L \dashv R)$$

between (elementary) topoi. A *structured cospan*, or the more emphatic term *L-structured cospan*, is a diagram in X of the form $La \to x \leftarrow Lb$.

Though structured cospans are defined abstractly, it is helpful to view the generic mathematical objects as networks. This 'network interpretation' leverages our intuition for physical systems. Interpret A as containing the interface types of a compositional network and X as containing the not–yet composable networks. The left adjoint L reinterprets each possible interface type $a \in \mathrm{ob}(\mathsf{A})$ as a degenerate network $La \in \mathrm{ob}(\mathsf{X})$. The right adjoint R returns the maximal subobject of a network that can serve as an interface. A structured cospan, then, is a diagram

$$\text{inputs} \to \text{network} \leftarrow \text{outputs}$$

That is, the apex of the cospan is a network and the arrows from the feet of the cospan determine the parts of the network that operate as inputs and outputs. The cospan in its entirety is a composable network.

Before discussing the operation of composing structured cospans, we first remark that the terms "inputs" and "outputs" do not connote causal structure. The sum of inputs and outputs of a network is its interface. The terms serve to partition the interface into two components that, when composing with another network, forms the connection and its complement.

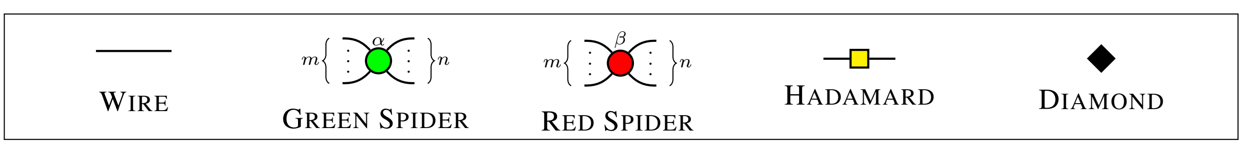


Figure 1: Generating diagrams for the ZX-calculus

To appear in EPTCS.

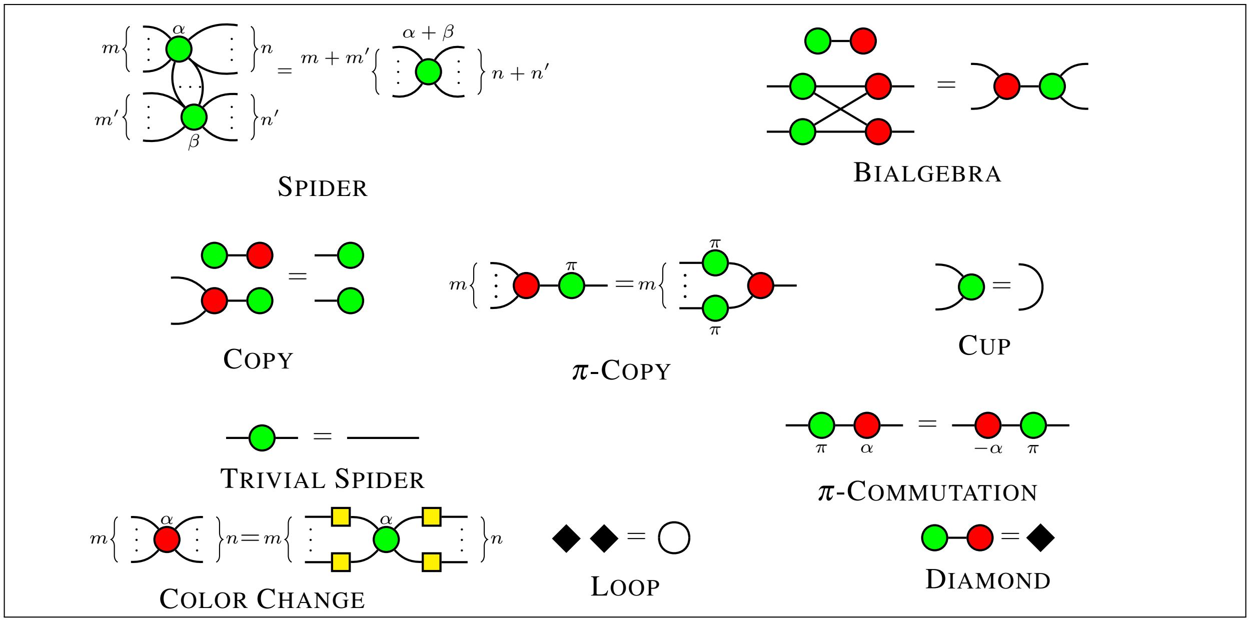


Figure 2: Relations in the ZX-calculus

Composition of structured cospans uses pushout. That is, the composite of the pair of structured cospans

$$La \to x \leftarrow Lb \quad \text{and} \quad Lb \to y \leftarrow Lc$$

is the structured cospan

$$La \to x +_{Lb} y \leftarrow Lc.$$

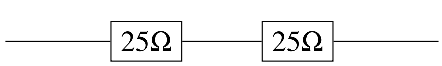
This operation has an intuitive meaning in the network interpretation. The network x has outputs Lb and the network y has inputs Lb. Because the inputs and outputs coincide, we can "connect y to x". The mechanism by which this connection happens is the pushout. The resulting network $x +_{Lb} y$ takes its inputs La from x and its outputs Lc from y. Thus, structured cospans model network composition.

To import rewriting to structured cospans, and therefore compositional networks, we use the double pushout formalism axiomatized with adhesive categories by Lack and Sobociński [4]. For our needs, the full generality of an adhesive category is excessive. Instead, we restrict our attention to elementary topoi, all of which are adhesive [5].

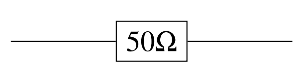
A *rewrite rule* is a span

$$\ell \leftarrow k \to r$$

in a topos T. While some authors require the arrows in the span to be monic, we do not. In the network interpretation, each object in a rewrite rule is a network. The rule states that the network ℓ is, in some sense, equivalent to the network r. This equivalence typically arises from two networks having the same semantics. An example of this is the semantic equivalence of the two distinct resistor circuits



and



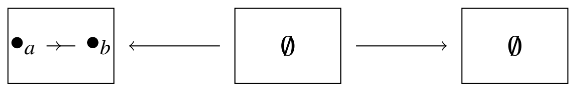
Packaging a rewrite rule into a span provides a usable device to identify an instance of ℓ in a larger network and replace that instance with r. The network k is the part of ℓ that is fixed through the rewriting process.

The act of replacing ℓ with r uses double pushout diagrams. That is, given a rewrite rule $\ell \leftarrow k \rightarrow r$ and a copy of ℓ inside another object g as encoded by an arrow $\ell \rightarrow g$, the existence of two conjoined pushout squares in T

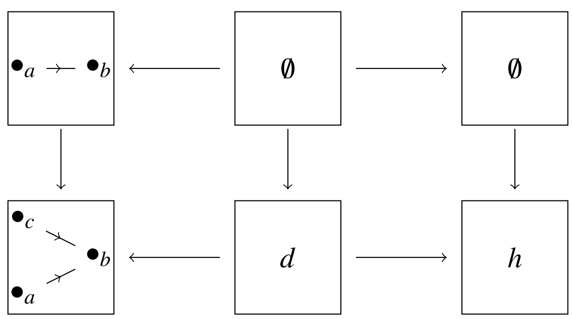
$$\begin{array}{ccccc} \ell & \longleftarrow & k & \longrightarrow & r \\ \downarrow & \ulcorner & \downarrow & \urcorner & \downarrow \\ g & \longleftarrow & d & \longrightarrow & h \end{array}$$

asserts that g rewrites into h. The object d is the subobject of g that is fixed during this rewrite.

The pushouts ensure that the rewrite is performed cohesively. When T is the topos of finite directed multi-graphs FinGraph, the cohesion imbued by the double pushouts has a concrete meaning. It ensures that the operation of applying a rewrite rule to a graph actually returns a graph. To exemplify what can go wrong, consider the rewrite rule



that deletes an edge along with its source and target. Let us try to apply this rule to a graph with another edge coinciding with a deleted node. It fails to produce a well-defined graph. The double pushout prevents such an occurrence. We present the following example: there do not exist graphs d and h that complete the double pushout diagram



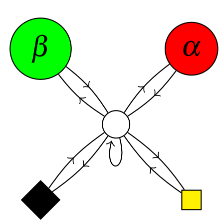
This cohesion identified in FinGraph abstracts to a general topos T.

Now, with structured cospans and rewriting each separately defined, we combine them to create a theory of rewriting structured cospans. The author studies this theory in a general fashion [2]. Here, we restrict our attention to fitting the ZX-calculus into this picture. First, we associate to each ZX-diagram a structured cospan. Second, we associate to each equation between ZX-diagrams a rewrite rule in the form of a span. The spans of structured cospans fit into a symmetric monoidal double category.

To capture the ZX-diagrams (Figure 1) as structured cospans, we start with a geometric morphism

$$\mathsf{FinGraph} \downarrow \Gamma \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{FinSet} \quad (L \dashv R)$$

The category $\mathsf{FinGraph} \downarrow \Gamma$ is the category of directed multi-graphs over

$\Gamma :=$ 

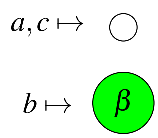
$\alpha, \beta \in [-\pi, \pi)$

For each value in $[-\pi, \pi)$, there is an associated red node and green node. When a green or red node is unlabeled, we take the label to be 0.

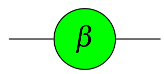
Graphs over Γ have the same typing information as the ZX-diagrams. This information is provided by the fibers of the map to Γ. For example, consider the graph

$$\bullet_a \rightarrowtail \bullet_b \rightarrowtail \bullet_c$$

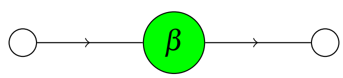
with the map to Γ determined by



This map carries the same syntactical information as the ZX-diagram



All ZX-diagrams can be translated into a graph over Γ. Note that the white node of Γ is a translation of the dangling strings of the ZX-diagrams. For convenience, we represent a graph over Γ by drawing the nodes of the graph in the same manner as those of Γ. For example, we draw the above graph over Γ as



It follows from the fundamental theorem of topos theory that, because $\mathsf{FinGraph}$ is a topos, $\mathsf{FinGraph} \downarrow \Gamma$ is also a topos. Thus, we have defined the domain and codomain of our geometric morphism. It remains to define L and R.

Define

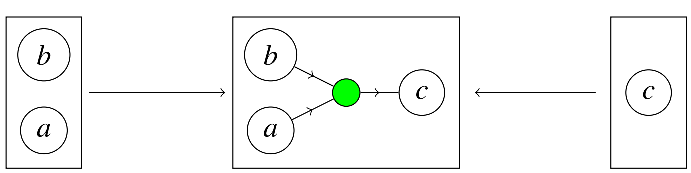
$$L \colon \mathsf{FinSet} \to \mathsf{FinGraph} \downarrow \Gamma$$

by letting La be the edgeless graph with node set a that is constant over Γ's white node. Define

$$R \colon \mathsf{FinGraph} \downarrow \Gamma \to \mathsf{FinSet}$$

by letting $R(x \to \Gamma)$ be the fiber in x of the white node. One can verify that this is indeed a geometric morphism.

Given an L-structured cospan $La \to x \leftarrow Lb$, both La and Lb are edgeless graphs with white nodes. The object x is a graph over Γ which has inputs and outputs those white nodes chosen by the arrows from La and Lb. For example



Now that we have structured cospans defined for the ZX-diagrams, we introduce their rewriting. This requires that structured cospans form a topos. Fortunately, this is the case.

Theorem. *Fix a geometric morphism $L \dashv R \colon \mathsf{X} \to \mathsf{A}$. Let ${}_L\mathsf{StrCsp}$ be the category whose objects are structured cospans and arrows are commuting diagrams*

$$\begin{array}{ccccc} La & \longrightarrow & x & \longleftarrow & Lb \\ {\scriptstyle Lf}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Lh} \\ La' & \to & x' & \leftarrow & Lb' \end{array}$$

Then ${}_L\mathsf{StrCsp}$ is a topos.

Because ${}_L\mathsf{StrCsp}$ is a topos, rewriting structured cospans enjoys the nice properties expected of a rewriting theory [4]. A *rewrite of structured cospans* is a connected component of diagrams in $\mathsf{FinGraph} \downarrow \Gamma$ of the form

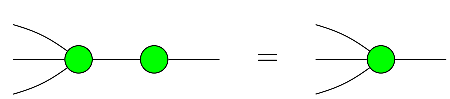
$$\begin{array}{ccccc} La & \longleftarrow & x & \longrightarrow & Lb \\ \cong\uparrow & & \uparrow & & \uparrow\cong \\ La' & \leftarrow & x' & \rightarrow & Lb' \\ \cong\downarrow & & \downarrow & & \downarrow\cong \\ La'' & \leftarrow & x'' & \rightarrow & Lb'' \end{array}$$

By connected component, we mean the equivalence relation generated by relating the above diagram to

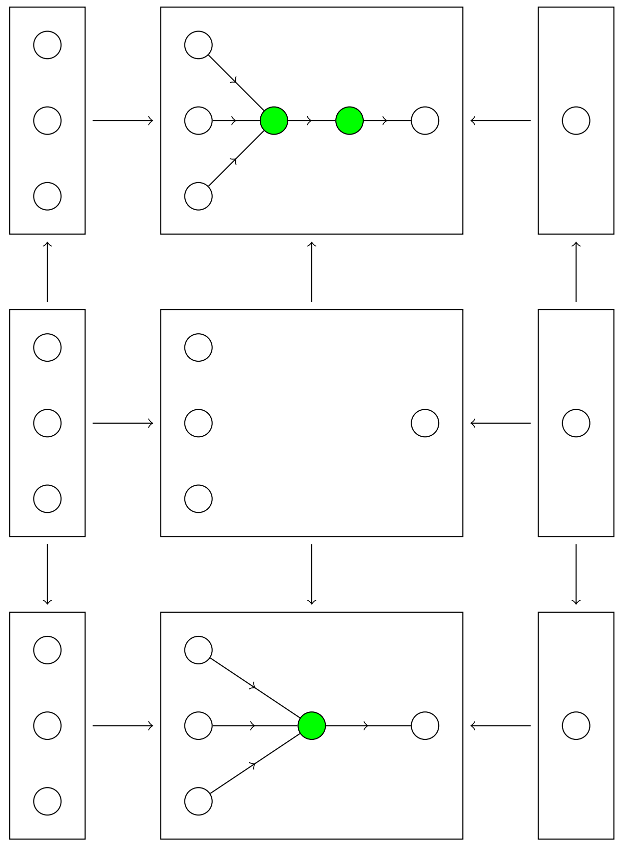
$$\begin{array}{ccccc} La & \longleftarrow & x & \longrightarrow & Lb \\ \cong\uparrow & & \uparrow & & \uparrow\cong \\ La' & \leftarrow & y' & \rightarrow & Lb' \\ \cong\downarrow & & \downarrow & & \downarrow\cong \\ La'' & \leftarrow & x'' & \rightarrow & Lb'' \end{array}$$

if there is an arrow $x' \to y'$ fitting into the evident commuting diagram.

We can encode each generating equation in the ZX-calculus into structured cospan rewrite. For example, to the ZX-diagram equation



we associate the structured cospan rewrite represented by



We replicated this for each generating equation of the ZX-calculus (see Figure 2).

Theorem. *There is a symmetric monoidal double category* ${}_L\mathbb{R}\mathsf{ewrite}$ *consisting of objects* $a \in \mathsf{ob}(\mathsf{FinSet})$*, vertical 1-arrows the L-structured cospans* $La \to x \leftarrow Lb$ *of graphs over* Γ*, horizontal 1-arrows spans in* FinSet *with invertible legs, and 2-arrows connected components of L-structured cospan rewrites. The tensor is given by pointwise application of the coproducts in* FinSet *and* $\mathsf{FinGraph} \downarrow \Gamma$*.*

Moreover, ${}_L\mathbb{R}\mathsf{ewrite}$ is isofibrant hence gives rise to a symmetric monoidal bicategory ${}_L\mathbf{Rewrite}$ with objects $a \in \mathsf{ob}(\mathsf{FinSet})$, 1-arrows are structured cospans $La \to x \leftarrow Lb$ of graphs over Γ, and 2-arrows are structured cospan rewrites of the form

$$\begin{array}{ccccc} La & \longleftarrow & x & \longrightarrow & Lb \\ {\scriptstyle =}\uparrow & & \uparrow & & \uparrow{\scriptstyle =} \\ La & \longleftarrow & x' & \longrightarrow & Lb \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow{\scriptstyle =} \\ La & \longleftarrow & x'' & \longrightarrow & Lb \end{array}$$

This fact follows from work by Shulman [6, Thm. 1.2].

Theorem. *The bicategory* ${}_L\mathbf{Rewrite}$ *is a bicategory of relations.*

The double category ${}_L\mathbb{R}\mathsf{ewrite}$ serves as an ambient space in which we can chisel out the structured cospans correlating to the ZX-calculus and their rewrites. Take the isofibrant symmetric monoidal double category $\mathbb{ZX}$ sitting inside ${}_L\mathbb{R}\mathsf{ewrite}$ that has finite sets as objects, the structured cospans corresponding to the ZX-diagrams generating the horizontal 1-arrows, the spans of finite sets with invertible legs as vertical 1-arrows, and the structured cospan rewrites that correspond to the ZX-diagram equations generating the 2-arrows. Then $\mathbb{ZX}$ encodes the ZX-calculus and its rewrites. The nature of this encoding is found in this following result.

Theorem. *Let* ZX *be the category whose objects are finite sets and arrows are the connected components of* **ZX***, the horizontal bicategory of* $\mathbb{ZX}$*. Then* ZX *is equivalent to the category whose objects are the natural numbers and arrows of type* $m \to n$ *are the ZX-diagrams with* m *inputs and* n *outputs modulo equations in Figure 2.*

The following is a nice related result.

Theorem. **ZX** *is a bicategory of relations.*

References

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