QUANTUM COMPUTING, SEIFERT SURFACES AND SINGULAR FIBERS

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Abstract. The fundamental group $\pi_1(L)$ of a knot or link L may be used to generate magic states appropriate for performing universal quantum computation and simultaneously for retrieving complete information about the processed quantum states. In this paper, one defines braids whose closure is the L of such a quantum computer model and computes their Seifert surfaces and the corresponding Alexander polynomial. In particular, some d-fold coverings of the trefoil knot, with $d = 3$, 4, 6 or 12, define appropriate links L and the latter two cases connect to the Dynkin diagrams of E_6 and D_4, respectively. In this new context, one finds that this correspondence continues with the Kodaira's classification of elliptic singular fibers. The Seifert fibered toroidal manifold Σ', at the boundary of the singular fiber $\tilde{E}_8$, allows possible models of quantum computing.

Summary

To acquire computational advantage over a classical circuit, a quantum circuit needs a non-stabilizer quantum operation for preparing a non-Pauli eigenstate, often called a magic state. The work about qubit magic state distillation [1] was generalized to qudits [2] and multi-qubits (see [3] for a review). Thanks to these methods, universal quantum computation (UQC), the ability to prepare every quantum gate, is possible. A new approach of UQC, based on permutation gates and simultaneously minimal informationally complete positive operator-valued measures (MICs), was worked out in [4, 5]. It is notable that the structure of the projective special linear group (or modular group) Γ is sufficient for getting most permutation-based magic states [6] used by us for UQC and that this can be thought of in terms of the complement of the trefoil knot in the 3-sphere S^3 [7].

Let us recall the context of our work compared to the existing literature. Bravyi & Kitaev introduced the principle of 'magic state distillation' [1]: universal quantum computation, the possibility of getting an arbitrary quantum gate, may be realized thanks to stabilizer operations (Clifford group unitaries, preparations and measurements) and an appropriate single qubit non-stabilizer state, called a 'magic state'. Then, irrespective of the dimension of the Hilbert space where the quantum states live, a non-stabilizer pure state was called a magic state [2]. An improvement of this concept was carried out in [4, 5] showing that a magic state could be at the same time a fiducial state for the construction of a minimal informationally complete positive operator-valued measure, or MIC, under the action on it of the Pauli

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group of the corresponding dimension. Thus UQC in this view happens to be relevant both to magic states and to MICs. In [4, 5], a d-dimensional magic state is obtained from the permutation group that organizes the cosets of a subgroup H of index d of a two-generator free group G. This is due to the fact that a permutation may be realized as a permutation matrix/gate and that mutually commuting matrices share eigenstates - they are either of the stabilizer type (as elements of the Pauli group) or of the magic type. It is enough to keep magic states that are simultaneously fiducial states for a MIC because the other magic states may loose the information carried during the computation. A catalog of the magic states relevant to UQC and MICs can be obtained by selecting G as the two-letter representation of the modular group $\Gamma = PSL(2,\mathbb{Z})$ [6]. The next step, developed in [7], is to relate the choice of the starting group G to three-dimensional topology. More precisely, G is taken as the fundamental group $\pi_1(S^3 \setminus L)$ of a 3-manifold M^3 defined as the complement of a knot or link L in the 3-sphere S^3. A branched covering of degree d over the selected M^3 has a fundamental group corresponding to a subgroup of index d of $\pi_1(M^3)$ and may be identified as a sub-manifold of M^3, the one leading to a MIC is a model of UQC. In the specific case of Γ, the knot involved is the left-handed trefoil knot $T_1 = 3^1$, as shown in Sec. [6] and [7, Sec. 2].

Motivation of the work. It is desirable that the UQC approach of [4]-[7] be formulated in terms of braid theory to allow a physical implementation. Braids of the anyon type, that are two-dimensional quasiparticles with world lines creating space-time braids, are nowadays very popular [8, 10, 9]. Close to this view of topological quantum computation (TQC) based on anyons, we propose a TQC based on Seifert surfaces defined over a link L. The links in question will be those able to generate magic states appropriate for performing permutation-based UQC.

In our previous work [7], we investigated the trefoil knot as a possible source of d-dimensional UQC models through its subgroups of index d (corresponding to d-fold coverings of the T_1 3-manifold) (see [7, Table 1]). More precisely, the link $L7n1$, corresponding to the congruence subgroup $\Gamma_0(2)$ of the modular group Γ, builds a relevant qutrit magic state for UQC whose MIC geometry is related to the Hesse configuration. The link $L6a3$, corresponding to the congruence subgroup $\Gamma_0(3)$ of Γ, builds a relevant two-qubit magic state whose MIC geometry is the generalized quadrangle of order two $GQ(2,2)$, as for the commutation of two-qubit Pauli operators. Then the link identified by the software SnapPy as $L6n1$ (or sometimes $L8n3$), corresponding to the congruence subgroup $\Gamma(2)$ of Γ, defines a 6-dit MIC with a building block geometry looking like Borromean rings [6, Fig. 4]. But none of the two links $L6n1$ and $L8n3$ is correctly associated to the subgroup $\Gamma(2)$ of Γ, but the link 6^3_3 (related to the Dynkin diagram of $\tilde{D}_4$) is. The possible confusion lies in the fact that all three links share the same link group $\pi_1(L)$. Finally, the link along Dynkin diagram of D_4 (with the icosahedral symmetry of H_3 in the induced permutations) is associated to a 12-dimensional (two-qubit/qutrit) MIC corresponding to the congruence subgroup $10A^1$ of Γ [6, Table 1].

Contents of the work. As announced in the abstract, we introduce a Seifert surface algorithm for converting the UQC models based on the aforementioned links into the appropriate braid representation permitted by Alexander's theorem [11]. A Seifert surface is an oriented surface whose boundary is a given link. Of course it is not unique. In this paper, to generate a Seifert surface, one makes use of the braid representation of the link. Since it has a skein relation different from the one obeyed by anyons this kind of topological quantum computation cannot be anyon-based. The skein relation in question is in terms of the Alexander polynomial instead of the Jones polynomial. Taking into account the observation that some of our UQC models are related to affine Coxeter-Dynkin diagrams, we build a class of UQC models starting with affine Dynkin diagrams of type $\tilde{D}_4$, $\tilde{E}_6$ and $\tilde{E}_8$, that are singular fibers of minimal elliptic surfaces. They are the precursors of 4-manifold topology that is currently under active scrutiny [12, 13].

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