Contextual advantage for state-dependent cloning

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arXiv:1905.08291

The no-cloning theorem

The no-cloning theorem states that there is no quantum channel $\mathcal{C}$ such that for a pair of distinct and nonorthogonal quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$,

$$\mathcal{C}(|\psi_i\rangle\langle\psi_i|) = |\psi_i\rangle\langle\psi_i| \otimes |\psi_i\rangle\langle\psi_i|\,, \quad i = 1, 2.$$

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  "operationally indistinguishable experimental procedures should be represented by the same object in the model."

No-cloning in noncontextual models

Noncontextual models for fragments of quantum theory admitting a no-cloning theorem:

- ERL mechanics, by Bartlett et al.[1]
- Spekkens' toy theory[2].

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So, we wonder:

is there any aspect of the phenomelogy of quantum state-cloning which cannot be explained by noncontextual models?

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Imperfect cloning

$|a\rangle$ and $|b\rangle$ sent with equal probability into a cloning machine $\mathcal{M}$. Look for $\mathcal{M}$ maximizing the global average fidelity

$$F_g^Q := \frac{1}{2}F(\mathcal{M}(|a\rangle\langle a|), |aa\rangle\langle aa|) + \frac{1}{2}F(\mathcal{M}(|b\rangle\langle b|), |bb\rangle\langle bb|).$$

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Bruß et al.[3] showed that the optimal quantum cloning fidelity is

$$F_g^{\mathrm{Q,opt}}(c_{ab}) := \frac{1}{4}\left[\sqrt{(1+c_{ab})(1+\sqrt{c_{ab}})} + \sqrt{(1-c_{ab})(1-\sqrt{c_{ab}})}\right]^2$$

where $c_{ab} := |\langle a|b\rangle|^2$.

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Main result y outline of the talk

Theorem (Informal)

The value of $F_g(c_{ab})$ *predicted by noncontextual models is below* $F_g^{\mathrm{Q,opt}}(c_{ab})$ *for all* $0 < c_{ab} < 1$*. In other words, contextuality provides an advantage for the task of imperfect cloning.*

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Talk's Outline:

1. Generalized noncontextuality.
2. Operational description of the cloning experiment.
3. Noncontextual bound for state-dependent cloning (main result). Proof idea and tightness.
4. Noise-robust version of the bound.
5. Conclusion and outlook.

Generalized noncontextuality

Ontological models for operational theories

Element	QT	Ontological models
Preparations P	Density Operators ρ	Ontic space Λ $\mu_P : \Lambda \to [0,1]$ $\mu_P(\lambda) \geq 0 \quad \forall\lambda$ $\int d\lambda \mu_P(\lambda) = 1$
Measurements M	POVMs $\{E_k\}$	$\xi_{k\|M}(\lambda) : \Lambda \to [0,1]$ $\sum_k \xi_{k\|M}(\lambda) = 1 \forall\lambda$
Transformations T	CPTP maps $\mathcal{E}$	$\Gamma_T : \Lambda_{in} \times \Lambda_{out} \to [0,1]$ $\int_{\Lambda_{out}} d\lambda_o \Gamma_T(\lambda_i, \lambda_o) = 1 \quad \forall\lambda_i$
$p(k\|M,P,T)$	$\mathrm{Tr}(\mathcal{E}(\rho))$	$\int d\lambda' d\lambda \xi_{k\|M}(\lambda')\Gamma_T(\lambda,\lambda')\mu_P(\lambda)$

Preparation noncontextuality

We say that two preparations P_1 and P_2 are *operationally equivalent* (denoted $P_1 \simeq P_2$) iff $p(k|M, P_1) = p(k|M, P_2)$ for all k and M.

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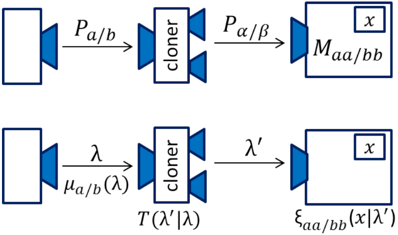
We say that an ontological model is *preparation noncontextual*[4] iff

$$P_1 \simeq P_2 \implies \mu_{P_1}(\lambda) = \mu_{P_2}(\lambda) \quad \forall \lambda$$

[4] Robert W Spekkens. "Contextuality for preparations, transformations, and unsharp measurements". In: *Physical Review A* 71.5 (2005), p. 052108.

Generalized noncontextuality

Cloning in operational terms



- Preparations P_s
  - Inputs: P_a and P_b.
  - Outputs: P_α and P_β.
  - Ideal clones: P_{aa} and P_{bb}.

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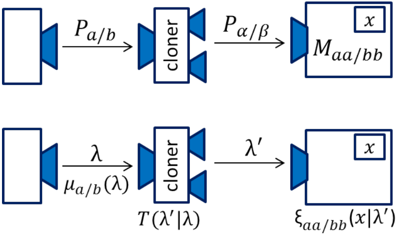

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- Test measurements M_s

$$p(s|P_s, M_s) = 1 \quad s \in \{a, b, \alpha, \beta, aa, bb\}.$$

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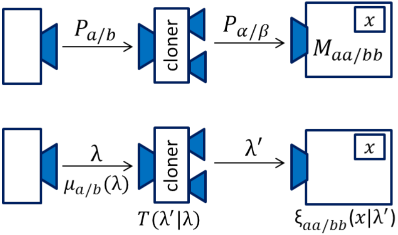

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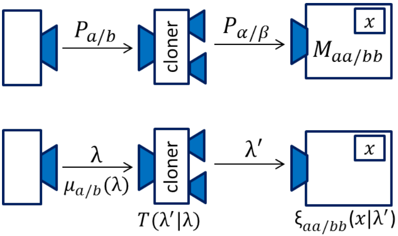

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- Note that $F_g := \frac{c_{\alpha aa} + c_{\beta bb}}{2}$
- Complementary preparations $P_{s^\perp}$

$$p(s|P_{s^\perp}, M_s) = 0 \quad s \in \{a, b, \alpha, \beta, aa, bb\},$$

$$\frac{1}{2}P_s + \frac{1}{2}P_{s^\perp} \simeq \frac{1}{2}P_{s'} + \frac{1}{2}P_{s'^\perp} \quad (s, s') \in \{(a, b), (aa, bb), (\alpha, aa), (\beta, bb)\}.$$

Operational description of state-dependent cloning

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In quantum mechanics:

- $P_a \leftarrow |a\rangle$ and $P_b \leftarrow |b\rangle$.
- $P_{a^\perp} \leftarrow |a^\perp\rangle$ and $P_{a^\perp} \leftarrow |a^\perp\rangle$ with $|a^\perp\rangle, |b^\perp\rangle \in \text{span}\{|a\rangle, |b\rangle\}$ and $\langle a|a^\perp\rangle = \langle b|b^\perp\rangle = 0$.
- $M_a \leftarrow \{|a\rangle\langle a|, \mathbb{I} - |a\rangle\langle a|\}$ and $M_b \leftarrow \{|b\rangle\langle b|, \mathbb{I} - |b\rangle\langle b|\}$.

Main result

Noncontextual bound on state-dependent cloning

Theorem

In any noncontextual ontological model satisfying for all (s, s') *in* $\{(a,b),(aa,bb),(\alpha,aa),(\beta,bb)\}$

1. $\frac{1}{2}P_s + \frac{1}{2}P_{s^\perp} \simeq \frac{1}{2}P_{s'} + \frac{1}{2}P_{s'^\perp}$,
2. $p(M_k|P_k) = 1, \quad p(M_k|P_{k^\perp}) = 0, \quad k = s, s'.$

we have that

$$F_g \leq F_g^{\mathrm{NC}} := \; 1 - \frac{c_{ab}}{2} + \frac{c_{aa,bb}}{2}$$

Proof ingredients:

- Lemma: $\|\mu_a - \mu_b\| = 2(1 - c_{ab})$ for models satisfying the Thm.'s hypothesis.

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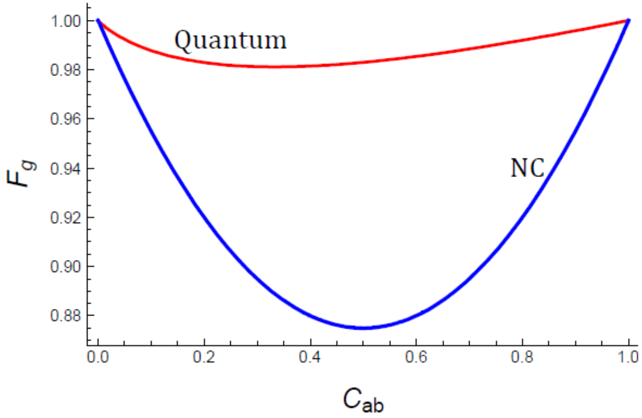
- $\|\cdot\|$ nonincreasing under stochastic maps and triangle inequality.

Quantum vs noncontextual cloning fidelities



$F_g^Q(c_{ab}) = \frac{1}{4}\left[\sqrt{(1+c_{ab})(1+\sqrt{c_{ab}})} + \sqrt{(1-c_{ab})(1-\sqrt{c_{ab}})}\right]^2$ (red line)

$F_g^{NC}(c_{ab}) = 1 - c_{ab}/2 + c_{ab}^2/2$ (blue line)

Tightness

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5. On the other hand, $\mu_\alpha(\lambda, \lambda') = \mu_a(\lambda)\mu_a(\lambda')$ for $\lambda \in S_a \backslash S_b$ and $\mu_\alpha(\lambda, \lambda') = \mu_a(\lambda)\mu_b(\lambda')$ for $\lambda \in S_a \cap S_b$.

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6. Finally,
$$F_g = \frac{1}{2}[c_{\alpha aa} + c_{\beta bb}] = \frac{1}{2}[1 - c_{ab} + c_{ab}^2] + \frac{1}{2} = F_g^{NC}.$$

Noise-robustness

Noise-robust noncontextual bound on state-dependent cloning

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2. $p(M_k|P_k) \geq 1 - \epsilon, \quad p(M_k|P_{k^\perp}) \leq \epsilon, \quad k = s, s'$.

we have that

$$F_g \leq F_g^{\mathrm{NC}} := 1 - \frac{c_{ab}}{2} + \frac{c_{aa,bb}}{2} + 8\epsilon$$

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Moreover, if, additionally, $c_{ss'} = c_{s's}$ *for all* (s, s') *in* $\{(a, b), (aa, bb), (\alpha, aa), (\beta, bb)\}$ *then*

$$F_g \leq F_g^{\mathrm{NC}} := 1 - \frac{c_{ab}}{2} + \frac{c_{aa,bb}}{2} + 2\epsilon$$

Tolerance to depolarizing noise

Consider the ideal quantum preparations, measurements and unitary transformation thwarted by a depolarizing channel $\mathcal{N}_v$:

$$\mathcal{N}_v(\rho) = (1 - v)\,\rho + v\mathbb{I}/4.$$

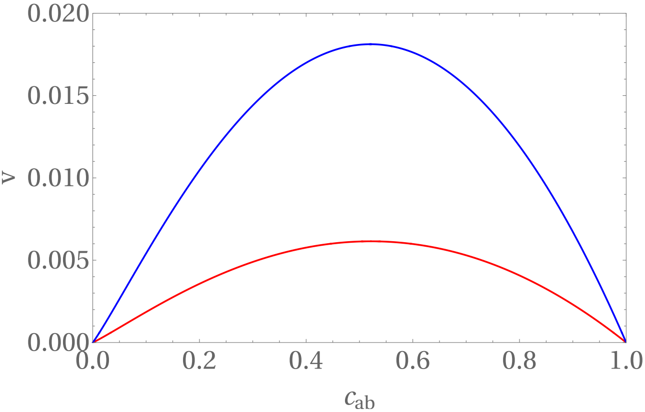

Figure: Max value of v for which there is a quantum violation of the NC bound as a function of c_{ab} (with and without the $c_{ab} = c_{ba}$ assumption).

General relation between ℓ_1 distance and confusability in NC models

Main ingredient in the proof of the noise-robust noncontextual bound:

Lemma

Let P_s, $P_{s'}$ be preparations. Suppose there exists preparations $P_{s^\perp}$, $P_{s'^\perp}$ and a two outcome measurement M_s such that

1. $\frac{1}{2}P_s + \frac{1}{2}P_{s^\perp} \simeq \frac{1}{2}P_{s'} + \frac{1}{2}P_{s'^\perp}$,
2. $p(M_k|P_k) \geq 1-\epsilon, \quad p(M_k|P_{k^\perp}) \leq \epsilon, \quad k = s, s'$.

Then, in a noncontextual ontological model,

$$2(1-c_{ss'}) - 8\epsilon \leq \|\mu_s - \mu_{s'}\| \leq 2(1-c_{ss'}) + 8\epsilon.$$

Moreover, if $c_{ss'} = c_{s's}$, then

$$2(1-c_{ss'}) - 2\epsilon \leq \|\mu_s - \mu_{s'}\| \leq 2(1-c_{ss'}) + 2\epsilon.$$

Conclusions and outlook

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  - For instance, it already provides an alternative proofs of the result from[5] (proven there via Fourier-Motzkin elimination).
- Future research: extend the results to universal and/or phase-covariant cloning and to probabilistic cloning.

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